Fano 2-352-75-a. $\mathscr{Z}\big(\mathbb{P}^3 \times \mathrm{Gr}(2,6),\ \mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee \oplus \mathcal{O}(1,1)\big)$.

Invariants. $h^0(-K) = 75,\ (-K)^4 = 352,\ h^{1,1} = 2,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 6,\ -\chi(T) = -4$.

Description.

(1) $\pi_1 \colon X \to \mathrm{Gr}(2,6)$ is an extremal small contraction onto a fourfold of degree 7: it contracts a plane with normal bundle $\mathcal{O}(-1)^{\oplus 2}$ to the unique singular point of the image, locally the cone over $\mathbb{P}^2 \times \mathbb{P}^1$, and is an isomorphism elsewhere.

(2) $\pi_2 \colon X \to \mathbb{P}^3$ is an extremal contraction of fiber type, with general fiber $\mathbb{P}^1$ and exactly four fibers isomorphic to $\mathbb{P}^2$.

- h, L denote the pullbacks of the hyperplane class of $\mathbb{P}^3$ and of the Plücker class.
- $\mathrm{Nef}(X) = \langle h, L\rangle$ is simplicial.
- The flip of π_1 replaces the plane by a line with normal bundle $\mathcal{O}(-1)^{\oplus 3}$.
- X is rational.

Semiorthogonal decompositions.

- From π_2 and [Orl, Proposition 2.10], $\mathrm{D}^{\mathrm{b}}(X)$ has a full exceptional collection of length $4+4+4 = 12$: two copies of $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3)$ and one exceptional object for each $\mathbb{P}^2$-fiber.

Explanation.

Write $\mathbb{P}^3 = \mathbb{P}(V_4)$ and $\mathrm{Gr}(2,6) = \mathrm{Gr}(2,V_6)$. The section of $\mathcal{Q}_{\mathbb{P}^3} \boxtimes \mathcal{U}^\vee$ is a general surjection $\psi \colon V_6 \to V_4$, with kernel W_2, and vanishes at (x,U) if and only if $\psi(U) \subset x$, that is $U \subset U_3(x) := \psi^{-1}(x)$, a 3-space containing W_2. The section of $\mathcal{O}(1,1)$ is a general bilinear form β on $V_4 \times \bigwedge^2 V_6$, and imposes $\beta(x, \bigwedge^2 U) = 0$. Thus

$$X = \{(x,U) :\ U \subset U_3(x),\ \beta(x, \textstyle\bigwedge^2 U) = 0\}.$$

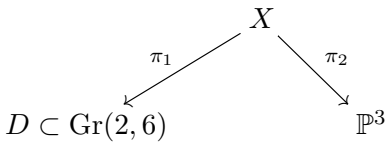
Description of π_1. The fiber of π_1 over U is $\{x : \psi(U) \subset x,\ \beta(x, \bigwedge^2 U) = 0\}$. It is empty if $\psi(U)$ has dimension 2. If $U \cap W_2$ is a line, then x is forced to be the point $\psi(U)$, and the fiber is this point if $\beta(\psi(U), \bigwedge^2 U) = 0$, and empty otherwise. Since $\psi|_U$ has constant rank one on $\{U : U \cap W_2 \text{ is a line}\}$, the map $U \mapsto (\psi(U), U)$ is a morphism there, inverse to π_1; so π_1 is an isomorphism outside the fiber over $[W_2]$. If $U = W_2$, the condition $\beta(x, \bigwedge^2 W_2) = 0$ is a hyperplane, so π_1 contracts exactly the plane $P = \{x : \beta(x, \bigwedge^2 W_2) = 0\} \times \{W_2\}$. The image D lies in $\{U : U \cap W_2 \neq 0\}$; near $[W_2]$, the two conditions on U cut a smooth sixfold of $\mathrm{Gr}(2,6)$, on which D is the rank-one locus of a general 3×2 matrix of local coordinates, that is the cone over $\mathbb{P}^2 \times \mathbb{P}^1$. So D has a unique singular point and is normal. Writing $\xi = L - h$ on the incidence $\{U \subset U_3(x)\}$, a $\mathrm{Gr}(2,3)$-bundle over $\mathbb{P}^3$, the projective bundle formula gives $\deg D = \int_X L^4 = 7$. A line $f_1 \subset P$ has $h \cdot f_1 = 1$ and $L \cdot f_1 = 0$, and the nef class L vanishes exactly on the curves of P; so π_1 is the contraction of an extremal ray, which is K_X-negative since X is Fano, and [Kaw] gives $N_{P/X} \simeq \mathcal{O}(-1)^{\oplus 2}$ and the flip.

Description of π_2. Over $x \in \mathbb{P}^3$, the planes $U \subset U_3(x)$ form $\mathbb{P}(U_3(x)^\vee) \simeq \mathbb{P}^2$, and the condition $\beta(x, \bigwedge^2 U) = 0$ is one linear condition on it, given by the restriction of $\beta(x, \cdot)$ to $\bigwedge^2 U_3(x)$. As $U_3(x) = W_2 \oplus \mathcal{O}(-1)$, we have $\bigwedge^2 U_3(x) = \mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2}$, so this restriction is a section (l, q_1, q_2) of $\mathcal{O}(1) \oplus \mathcal{O}(2)^{\oplus 2}$, with l linear and q_1, q_2 quadratic, general by the surjectivity of the restriction of sections. Hence X is the projectivization over $\mathbb{P}^3$ of the cokernel of $(l, q_1, q_2)\colon \mathcal{O}(-2) \to \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)$: its fibers are lines, except for planes over the four reduced points $Z = \{l = q_1 = q_2 = 0\}$. A line f_2 in a fiber has $L \cdot f_2 = 1$ and $h \cdot f_2 = 0$, and the nef class h vanishes exactly on the curves of the fibers. Since π_2 is generically a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$, X is rational.

Cones. The classes h, L are nef, and f_1, f_2 form the dual basis:

	f_1	f_2
h	1	0
L	0	1

Hence $\mathrm{Nef}(X) = \langle h, L \rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2 \rangle$ are dual simplicial cones, and π_1, π_2 are the two extremal contractions.



◇

References

[Kaw] Y. Kawamata, *Small contractions of four-dimensional algebraic manifolds*, Math. Ann. 284 (1989), 595–600.

[Orl] D. Orlov, *Triangulated categories of singularities and equivalences between Landau–Ginzburg models*, Sb. Math. 197 (2006), 1827–1840.