Fano 2-352-75-d. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathrm{Gr}(2,4),\ \mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{O}_{\mathrm{Gr}}(1) \oplus \mathcal{O}_{\mathbb{P}^2_1}(1) \boxtimes \mathcal{Q}_{\mathbb{P}^2_2}\big)$.

Invariants. $h^0(-K) = 75,\ (-K)^4 = 352,\ h^{1,1} = 2,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 4,\ -\chi(T) = -6$.

Description.

(1) $\pi_1 \colon X \to \mathbb{P}^2 \times \mathrm{Gr}(2,4)$ is an isomorphism onto $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2,4), \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}(1)\big)$.
(2) $\pi_2 \colon X \to \mathbb{P}^2 \times \mathrm{Gr}(2,4)$ is an isomorphism onto $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2,4), \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}(1)\big)$.
(3) $\pi_3 \colon X \to \mathbb{P}^2 \times \mathbb{P}^2$ is an extremal contraction of fiber type onto the graph of an isomorphism $\mathbb{P}^2 \simeq \mathbb{P}^2$. It is the quadric surface fibration induced by the linear projection of the quadric $Q^4 = \mathrm{Gr}(2,4)$ from a general plane; its discriminant is a smooth conic, over which the fibers are quadric cones.
(4) The projection $\pi_{12} \colon X \to \mathrm{Gr}(2,4)$, forgetting the first two factors, is an extremal divisorial contraction, the blow-up of Q^4 (1-512-105-a) along a smooth conic, its intersection with a general plane.

- h, L denote the pullbacks of the hyperplane class of $\mathbb{P}^2$ and of the Plücker class.
- $\mathrm{Nef}(X) = \langle h, L\rangle$ is simplicial.
- $\mathrm{Aut}(X)$ is the stabilizer of the conic in $\mathrm{Aut}(Q^4)$, of dimension 6.
- X is rational.

Semiorthogonal decompositions.

- From π_{12}, [Orl] and [Kap]: $\mathrm{D^b}(X) = \langle \mathrm{D^b}(C), \mathrm{D^b}(C), \pi_{12}^* \mathrm{D^b}(Q^4)\rangle$, where C is the conic; this is a full exceptional collection of length $2+2+6 = 10$.
- From π_3 and [Kuz], with $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ the double cover branched along the discriminant conic:

$$\mathrm{D^b}(X) = \langle \mathrm{D^b}(\mathbb{P}^1 \times \mathbb{P}^1),\ \pi_3^* \mathrm{D^b}(\mathbb{P}^2) \otimes \mathcal{O}(L),\ \pi_3^* \mathrm{D^b}(\mathbb{P}^2) \otimes \mathcal{O}(2L)\rangle.$$

Explanation.

Write $\mathbb{P}^2_1 = \mathbb{P}(V_3')$, $\mathbb{P}^2_2 = \mathbb{P}(V_3)$ and $\mathrm{Gr}(2,4) = \mathrm{Gr}(2,V_4) = Q^4 \subset \mathbb{P}(\bigwedge^2 V_4)$, with quadratic form q. The section of $\mathcal{Q}_{\mathbb{P}^2_2} \boxtimes \mathcal{O}_{\mathrm{Gr}}(1)$ is a general linear map $\alpha \colon \bigwedge^2 V_4 \to V_3$, and vanishes at (z, U) if and only if the composite $\bigwedge^2 U \to \bigwedge^2 V_4 \xrightarrow{\alpha} V_3 \to V_3/z$ vanishes, that is, $\alpha(\bigwedge^2 U) \subset z$. The section of $\mathcal{O}(1) \boxtimes \mathcal{Q}_{\mathbb{P}^2_2}$ is a general isomorphism $\beta \colon V_3' \to V_3$, and imposes $\beta(x) \subset z$. Let $K = \ker \alpha$, so that $\Pi = \mathbb{P}(K)$ is a general plane, $q|_K$ is nondegenerate, and $C = Q^4 \cap \Pi$ is a smooth conic.

Description of π_1 and π_2. Since β is an isomorphism, $\beta(x) \subset z$ means $z = \beta(x)$: the zero locus of $\mathcal{O}(1) \boxtimes \mathcal{Q}$ in $\mathbb{P}^2 \times \mathbb{P}^2$ is the graph of β [BFMT, Proposition 3.3]. So each of x, z determines the other, π_1 and π_2 are isomorphisms onto their images, and

$$X \simeq \mathscr{Z}\big(\mathbb{P}^2 \times Q^4,\ \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}(1)\big).$$

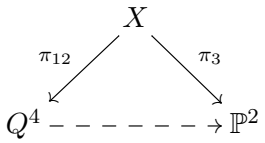
Description of π_{12}. By [BFMT, Lemma 3.1], applied with $m = 2$ and the hyperplane bundle of Q^4, π_{12} is the blow-up of Q^4 along the zero locus of the three components of α, that is, $Q^4 \cap \Pi = C$. A line e in a fiber $\mathbb{P}^2$ of the exceptional divisor has $h \cdot e = 1$ and $L \cdot e = 0$.

Description of π_3. Since π_1 is an isomorphism onto its image, π_3 is the projection $X \to \mathbb{P}^2_2$, and its fiber over z is $Q^4 \cap \Lambda_z$, where $\Lambda_z = \mathbb{P}(\alpha^{-1}(z)) \simeq \mathbb{P}^3$ contains Π. As $q|_K$ is nondegenerate, $\bigwedge^2 V_4 = K \oplus K^\perp$, and α identifies $K^\perp$ with V_3; so $\alpha^{-1}(z) = K \oplus \langle w_z \rangle$ with $w_z \in K^\perp$, and $q|_{\alpha^{-1}(z)} = q|_K \oplus q(w_z)$. Therefore every fiber is a quadric surface of rank at least 3, and the discriminant is the smooth conic $\{q|_{K^\perp}(w_z) = 0\} \subset \mathbb{P}^2$, over which the fibers are quadric cones. A line ℓ in a fiber has $h \cdot \ell = 0$ and $L \cdot \ell = 1$. Each fiber $\{c\} \times \mathbb{P}^2$ of the exceptional divisor of π_{12} is a section of π_3; hence the even Clifford algebra of π_3 has trivial Brauer class on the double cover of $\mathbb{P}^2$ branched along the discriminant, which is $\mathbb{P}^1 \times \mathbb{P}^1$, and [Kuz] gives the second decomposition.

Cones and automorphisms. The classes h and L are nef, and e, ℓ form the dual basis:

	e	ℓ
h	1	0
L	0	1

Hence $\mathrm{Nef}(X) = \langle h, L \rangle$ and $\overline{\mathrm{NE}}(X) = \langle e, \ell \rangle$ are dual simplicial cones, contracted by π_{12} and π_3 respectively.



Since the two contractions are of different types, every automorphism of X preserves π_{12}, so $\mathrm{Aut}(X)$ is the stabilizer of C in $\mathrm{Aut}(Q^4)$, that is, of the nondegenerate plane Π; its identity component is $\mathrm{SO}(K) \times \mathrm{SO}(K^\perp)$, of dimension 6. Finally, X is rational, being birational to Q^4.

◇

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