Fano 2-378-80-a. $\mathscr{Z}\big(\mathrm{Fl}(1,5,6),\ (\mathcal{U}_1^\vee)^{\oplus 4} \oplus \det(\mathcal{U}_5/\mathcal{U}_1)^{\vee\,2} \otimes \mathcal{U}_1^{\vee\,2}\big)$.

Invariants. $h^0(-K) = 80,\ (-K)^4 = 378,\ h^{1,1} = 2,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 4,\ -\chi(T) = -7$.

Description. We regard $\mathrm{Fl}(1,5,V_6) \subset \mathbb{P}(V_6) \times \mathrm{Gr}(5,V_6) = \mathbb{P}^5 \times \check{\mathbb{P}}^5$, in this order, with $\mathcal{U}_1 \subset \mathcal{U}_5$. The sections are:

- four linear forms, for $\mathcal{U}_1^\vee$, cutting out a plane $V_2 \subset V_6$ and imposing $u \in V_2$;
- a quadratic form Q on $V_6^\vee$, for the last summand, which equals $(\det \mathcal{U}_5^\vee)^2 = \mathcal{O}_{\check{\mathbb{P}}^5}(2)$. It imposes $U_5 = \ker \xi$ with $Q(\xi) = 0$.

Thus, with $\mathbb{P}^1 = \mathbb{P}(V_2)$ and $Q^4 = \{Q = 0\} \subset \check{\mathbb{P}}^5$,

$$X = \{(u,\xi) \in \mathbb{P}^1 \times Q^4 :\ \xi(u) = 0\},$$

a divisor of bidegree $(1,1)$ in $\mathbb{P}^1 \times Q^4$. Let $S = Q^4 \cap \mathbb{P}(V_2^\perp)$, a smooth quadric surface (a codimension two linear section of Q^4).

(1) $\pi_1 \colon X \to \mathrm{Gr}(5,6) = \check{\mathbb{P}}^5$ maps onto Q^4 (type 1-512-105-a), and $X = \mathrm{Bl}_S Q^4$. It is a K-negative extremal divisorial contraction with exceptional divisor $E = H - h \simeq S \times \mathbb{P}^1$.

(2) $\pi_2 \colon X \to \mathbb{P}^5$ maps onto the line $\mathbb{P}(V_2) \simeq \mathbb{P}^1$. It is the quadric fibration given by the pencil of hyperplane sections $Q^4 \cap \{\xi(u) = 0\}$ through S, an extremal contraction of fiber type. Its general fiber is a smooth quadric threefold, and two fibers are quadric cones with vertex outside S.

- Notation: $h = c_1(\mathcal{U}_1^\vee)$ and $H = c_1(\mathcal{U}_5^\vee)$, restricted to X; then $-K_X = h + 3H = 4H - E$.
- $\mathrm{Nef}(X) = \langle h, H\rangle$ is simplicial, and $\overline{\mathrm{NE}}(X) = \langle e, \ell\rangle$, where e is a fiber of $E \to S$ and ℓ is a line in a fiber of π_2.
- $\mathrm{Aut}(X)$ is the stabilizer of V_2 in $\mathrm{SO}(Q)$, of dimension $1 + 6 = 7 = \chi(T_X)$.
- Rationality: yes.

Semiorthogonal decompositions.

- From π_1 (Orlov and Kapranov): $\mathrm{D^b}(X) = \langle \mathrm{D^b}(S), \pi_1^* \mathrm{D^b}(Q^4)\rangle$, a full exceptional collection of length $4 + 6 = 10 = \chi_{\mathrm{top}}(X)$.
- From π_2 (Kuznetsov, quadric fibrations of odd relative dimension): $\mathrm{D^b}(X) = \langle \mathrm{D^b}(\mathbb{P}^1, \mathcal{B}_0),\ \pi_2^* \mathrm{D^b}(\mathbb{P}^1)(H),\ \pi_2^* \mathrm{D^b}(\mathbb{P}^1)(2H),\ \pi_2^* \mathrm{D^b}(\mathbb{P}^1)(3H)\rangle$.

Explanation.

Setup. We have $\det(\mathcal{U}_5/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee = \det \mathcal{U}_5^\vee$, so the last summand is $\mathcal{O}_{\check{\mathbb{P}}^5}(2)$, whose sections are the quadratic forms on $V_6^\vee$. The four sections of $\mathcal{U}_1^\vee$ descend to $\mathbb{P}^5$ and cut out the line $\mathbb{P}(V_2)$. Since $\mathrm{Fl}(1,5,6)$ is the incidence divisor $\{\xi(u) = 0\}$ of bidegree $(1,1)$ in $\mathbb{P}^5 \times \check{\mathbb{P}}^5$, we obtain the description of X as a $(1,1)$-divisor in $\mathbb{P}^1 \times Q^4$. For general data, Q^4 is smooth and $\mathbb{P}(V_2^\perp)$ is a general $\mathbb{P}^3$, so S is a smooth quadric surface. Moreover

$$-K_X = (2h + 4H) - (h + H) = h + 3H.$$

In $\mathbb{P}^1 \times Q^4$ with $[X] = h + H$, we have $hH^3 = 2$ and $H^4 = 2$. Hence

$$(-K_X)^4 = (81 + 4 \cdot 27) \cdot 2 = 378,$$

in agreement with FanoBase.

Description of π_1. Write the equation as $t_0\,\xi(v_0) + t_1\,\xi(v_1)$, for a basis v_0, v_1 of V_2. This is the setting of [BFMT, §2, first Cayley lemma] with $m = 1$ and $\mathcal{L} = \mathcal{O}_{Q^4}(1)$, whence

$$X = \mathrm{Bl}_Z\, Q^4, \qquad Z = \{\xi(v_0) = \xi(v_1) = 0\} \cap Q^4 = S.$$

Concretely, the fiber over ξ is the point $\ker\xi \cap V_2$ unless $V_2 \subset \ker\xi$, i.e. $\xi \in S$, in which case it is $\mathbb{P}^1$. Then $E \simeq S \times \mathbb{P}^1$, $-K_{Q^4} = 4H$ and $E = 4H - (h + 3H) = H - h$.

Description of π_2. For $p\colon \mathbb{P}^1 \times Q^4 \to \mathbb{P}^1$, we have $p_*\mathcal{O}(1,1) = V_6 \otimes \mathcal{O}(1)$, and the section is the composite

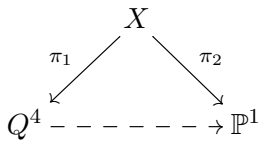
$$\mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{(v_0, v_1)} V_6 \otimes \mathcal{O}.$$

So the fiber over u is the hyperplane section $Q^4 \cap \{\xi(u) = 0\}$, of the pencil of hyperplanes containing $\mathbb{P}(V_2^{\perp}) \supset S$. It is singular exactly when the hyperplane is tangent to Q^4, i.e. when $[u]$ lies on the dual quadric $Q^\vee \subset \mathbb{P}^5$. The line $\mathbb{P}(V_2)$ meets $Q^\vee$ in two points, so two fibers are quadric cones. Their vertices are the poles of the corresponding hyperplanes, and they lie off S. The relative Picard number is one, and $-K \cdot \ell = 3$ for a line ℓ in a fiber.

Cones. The classes h and H are nef, being pulled back by π_2 and π_1, and

	e	ℓ
h	1	0
H	0	1
$E = H - h$	-1	1
$-K$	1	3

Hence $\mathrm{Nef}(X) = \langle h, H\rangle$ and $\overline{\mathrm{NE}}(X) = \langle e, \ell\rangle$ are dual simplicial cones.



Hodge numbers and automorphisms. From π_1, $h^{2,2} = 2 + b_2(S) = 4$ and $\chi_{\mathrm{top}} = 6 + 4 = 10$. An automorphism preserves both contractions, hence the pair (Q^4, S), i.e. the nondegenerate plane V_2. The stabilizer of V_2 in SO_6 is $S(\mathrm{O}_2 \times \mathrm{O}_4)$, of dimension $1 + 6 = 7$, which agrees with $\chi(T_X) = 7$.

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