Fano 2-384-81-a. $\mathscr{Z}\big(\mathrm{Fl}(1,3,5),\, \mathcal{U}_1^\vee \oplus S^2(V_5/\mathcal{U}_3)^\vee \otimes (\det \mathcal{U}_3^\vee)^2\big)$.

Invariants. $h^0(-K) = 81, \quad (-K)^4 = 384, \quad h^{1,1} = 2, \quad h^{2,2} = 2, \quad -\chi(T) = -10.$

Description. Regard $\mathrm{Fl}(1,3,5)$ as the incidence subvariety of $\mathbb{P}(V_5) \times \mathrm{Gr}(3,5)$, and let π_i forget the i-th factor. Set $H = c_1(\mathcal{U}_1^\vee|_X)$ and $b = c_1(\det \mathcal{U}_3^\vee|_X)$.

(1) $\pi_1 : X \to \mathrm{OG}(2,5) \simeq \mathbb{P}^3$ is a K-negative extremal $\mathbb{P}^1$-bundle. In fact, $X \simeq \mathbb{P}_{\mathbb{P}^3}(\mathcal{N}(-1))$, with $\mathcal{N}$ a null-correlation bundle, and $X \simeq \mathrm{Sp}_4/B$.

(2) $\pi_2 : X \to \mathbb{P}(V_4) \simeq \mathbb{P}^3 \subset \mathbb{P}(V_5)$, for a hyperplane $V_4 \subset V_5$, has Stein factorization

$$X \xrightarrow{\varpi} \mathrm{LG}(2,4) \simeq Q^3 \xrightarrow{2:1} \mathbb{P}^3.$$

Here ϖ is the second K-negative extremal $\mathbb{P}^1$-bundle, and the finite map is branched along a smooth quadric surface.

- Let h and k denote the pullbacks of the hyperplane classes of $\mathbb{P}^3$ and Q^3. Then $b = 2h$, $H = k$, $-K_X = 2h + 2k$, and $\mathrm{Nef}(X) = \langle h, k\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions. By Orlov's projective bundle formula [1],

$$\mathrm{D}^{\mathrm{b}}(X) = \big\langle \pi_1^*\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3),\, \pi_1^*\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3) \otimes \mathcal{O}_X(H)\big\rangle.$$

In particular, $\mathrm{D}^{\mathrm{b}}(X)$ admits a full exceptional collection of length eight.

Explanation. Let $\mathcal{Q} = V_5/\mathcal{U}_3$, of rank two. The identity $\mathcal{Q}^\vee \otimes \det \mathcal{Q} \simeq \mathcal{Q}$ identifies the second summand of the defining bundle with $S^2\mathcal{Q}$. A general section corresponds to a nondegenerate quadratic form on $V_5^\vee$. Its zero locus on $\mathrm{Gr}(3,5)$ is $Y = \mathrm{OG}(2,5) \simeq \mathbb{P}^3$: it parametrizes the three-spaces $U_3 = U_2^{\perp_q}$, where U_2 is a maximal isotropic plane for the inverse quadratic form q on V_5. The first summand is a linear form $\varphi \in V_5^\vee$, imposing $U_1 \subset V_4 = \ker \varphi$. Since $-K_{\mathrm{Fl}(1,3,5)} = 4b + 3H$ and the defining bundle has first Chern class $H + 3b$, adjunction gives $-K_X = b + 2H$.

Description of π_1. For the natural projection $p : \mathrm{Fl}(1,3,5) = \mathbb{P}_{\mathrm{Gr}(3,5)}(\mathcal{U}_3) \to \mathrm{Gr}(3,5)$, we have

$$p_*\mathcal{U}_1^\vee = \mathcal{U}_3^\vee, \qquad p_*p^*S^2\mathcal{Q} = S^2\mathcal{Q},$$

with vanishing higher direct images. Thus the first section restricts on Y to a morphism $\mathcal{U}_3|_Y \to \mathcal{O}_Y$, and X is the projectivization of its kernel wherever this morphism is surjective.

We identify the kernel using the spinor model. Let W be four-dimensional, with a nondegenerate alternating form ω, and identify V_5 with $\omega^\perp \subset \bigwedge^2 W^\vee$ for the nondegenerate wedge pairing. Under $Y \simeq \mathbb{P}(W)$, a line $\ell \subset W$ corresponds to the isotropic plane

$$U_2(\ell) = \bigwedge^2 \mathrm{ann}(\ell) \cap \omega^\perp \subset V_5.$$

The three-space $\bigwedge^2 \mathrm{ann}(\ell)$ is maximal isotropic for the wedge pairing on $\bigwedge^2 W^\vee$. Consequently its orthogonal in $\bigwedge^2 W^\vee$ equals itself, and $U_2(\ell)^{\perp_q} \subset V_5$ is the image of $\bigwedge^2 \mathrm{ann}(\ell)$ under orthogonal projection along ω. This projection is an isomorphism on each such three-space, since ω is nondegenerate. Hence

$$\mathcal{U}_3|_Y \simeq \bigwedge^2 \Omega^1_{\mathbb{P}^3}(1) \simeq T_{\mathbb{P}^3}(-2).$$

Let $\eta \in V_5$ be the q-dual of φ. For general φ, the alternating form η on W is nondegenerate. The induced morphism $T_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3}$ has, at $[\ell]$, the form

$$\ell \otimes (W/\ell) \longrightarrow \mathbb{C}, \qquad \ell \otimes \overline{w} \longmapsto \eta(\ell, w).$$

It is surjective at every point, since η is nondegenerate, and its kernel is $\ell \otimes (\ell^{\perp_\eta}/\ell)$. The rank-two bundle $\mathcal{N}_\eta$ with fibers $\ell^{\perp_\eta}/\ell$ is the null-correlation bundle, characterized by

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\eta} \Omega^1_{\mathbb{P}^3}(1) \longrightarrow \mathcal{N}_\eta \longrightarrow 0.$$

Thus $X \simeq \mathbb{P}_{\mathbb{P}^3}(\mathcal{N}_\eta(-1))$. A line in $\ell^{\perp_\eta}/\ell$ determines a unique η-Lagrangian plane Λ containing ℓ. This identifies X with the complete isotropic flag variety

$$\{(\ell, \Lambda) : \ell \subset \Lambda \subset W,\ \Lambda \in \mathrm{LG}_\eta(2, W)\} = \mathrm{Sp}_4/B.$$

Description of π_2. The projection $\varpi : (\ell, \Lambda) \mapsto \Lambda$ is a $\mathbb{P}^1$-bundle over $\mathrm{LG}_\eta(2, W) \simeq Q^3$. To recover the original map π_2, represent a Lagrangian plane Λ by a decomposable form $\beta \in \bigwedge^2 W^\vee$ with $\ker \beta = \Lambda$. Since Λ is η-Lagrangian, $\beta \wedge \eta = 0$. The corresponding point of $\mathbb{P}(V_5)$ is

$$u = \beta - \frac{\beta \wedge \omega}{\omega \wedge \omega}\,\omega \;\in\; V_5 \cap \eta^\perp = V_4.$$

Indeed, $\ell \subset \ker \beta$ implies $\beta \in \bigwedge^2 \mathrm{ann}(\ell)$, so the projection along ω places u in $U_2(\ell)^{\perp_q} = U_3(\ell)$; the condition $\beta \wedge \eta = 0$ places it in V_4. Thus $[u]$ is exactly the line U_1 of the original flag. Moreover $u \neq 0$, since a decomposable form cannot be a multiple of the nondegenerate ω.

It follows that π_2 is the composition of ϖ with the linear projection of $Q^3 \subset \mathbb{P}(\eta^\perp) \simeq \mathbb{P}^4$ from $[\omega] \notin Q^3$ onto $\mathbb{P}(V_4)$. Writing $\beta = u + t\omega$, the equation $\beta \wedge \beta = 0$ becomes

$$u \wedge u + t^2 \omega \wedge \omega = 0.$$

Hence the projection is a finite double cover, branched along $B = \{[u] \in \mathbb{P}(V_4) : u \wedge u = 0\}$. The wedge pairing is nondegenerate on V_5, and $\eta \in V_5$ is nonisotropic. Its restriction to $V_4 = \eta^\perp \cap V_5$ is therefore nondegenerate, so B is a smooth quadric surface. In particular, a general fiber of π_2 is the disjoint union of two copies of $\mathbb{P}^1$; the connected-fiber contraction is ϖ, not π_2.

The Plücker line bundle of $\mathrm{OG}(2, 5) \simeq \mathbb{P}^3$ is $\mathcal{O}_{\mathbb{P}^3}(2)$, so $b = 2h$. The construction above is linear in the Plücker coordinate $[\beta]$ of Λ, whence $H = k$. As both h and k are pullbacks of ample classes, are independent, and vanish on the fibers of the respective $\mathbb{P}^1$-bundles, they span $\mathrm{Nef}(X)$ for $\rho(X) = 2$. Finally, $-K_X = 2h + 2k$ has degree two on both extremal rays, and the projective bundle model proves rationality. ◇

References

[1] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.