Fano 2-416-87-a. $\mathscr{Z}\big(\mathrm{Fl}(1,3,6)\times\mathbb{P}^3,\ \mathcal{Q}^{\oplus 2}\oplus\mathcal{Q}\boxtimes\mathcal{O}_{\mathbb{P}^3}(1)\oplus\det(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes(\mathcal{U}_1^\vee)^{\otimes 2}\big)$, where $\mathcal{Q}=\bigwedge^2(V_6/\mathcal{U}_3)^\vee\otimes\det\mathcal{U}_3^\vee\simeq V_6/\mathcal{U}_3$.

Invariants. $h^0(-K)=87,\quad (-K)^4=416,\quad h^{1,1}=2,\quad h^{2,2}=4.$

Description. Regard $\mathrm{Fl}(1,3,6)$ as the incidence variety in $\mathbb{P}(V_6)\times\mathrm{Gr}(3,V_6)$, with this factor followed by $\mathbb{P}^3$; π_i forgets the i-th factor. Put $H=c_1(\mathcal{U}_1^\vee)|_X$ and $b=c_1(\det\mathcal{U}_3^\vee)|_X$. Let $Q^4\subset\mathbb{P}^5$ be the smooth quadric fourfold 1-512-105-a.

(1) $\pi_1\colon X\to\Gamma\simeq\mathbb{P}^3$ is an extremal contraction of fibre type, where $\Gamma\subset\mathrm{Gr}(3,V_6)\times\mathbb{P}^3$ is its image (the graph of an isomorphism). Its general fibre is $\mathbb{P}^1$, with exactly two fibres $\mathbb{P}^2$.

(2) The projection $\pi_{23}\colon X\to Q^4\subset\mathbb{P}^5$, forgetting the last two factors, is the blow-up of a line $\ell\subset Q^4$.

- $\mathrm{Nef}(X)=\langle H,b\rangle$ and $\overline{\mathrm{NE}}(X)=\langle[e],[f]\rangle$ are simplicial, where e is a line in a fibre of the exceptional divisor over ℓ, and f is a general fibre of π_1.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [2] gives

$$D^{\mathrm{b}}(X)=\big\langle D^{\mathrm{b}}(\ell),D^{\mathrm{b}}(\ell),\pi_{23}^*D^{\mathrm{b}}(Q^4)\big\rangle.$$

If $j\colon E\hookrightarrow X$ is the exceptional divisor and $p\colon E\to\ell$ its projection, the fully faithful embeddings of the first two components are $j_*(p^*(-)\otimes\mathcal{O}_E(-2))$ and $j_*(p^*(-)\otimes\mathcal{O}_E(-1))$, respectively; the last one is π_{23}^*.

Explanation. Write $V=V_6$ and let A be a four-dimensional vector space, so that the original ambient space is $\mathrm{Fl}(1,3,V)\times\mathbb{P}(A)$. Since $\det(V/\mathcal{U}_3)=\det\mathcal{U}_3^\vee$ (up to a constant line), $\mathcal{Q}\simeq V/\mathcal{U}_3$. The first two sections of $\mathcal{Q}$ give independent vectors $v_1,v_2\in V$, forcing $W=\langle v_1,v_2\rangle\subset U_3$. The mixed section is the composite

$$\mathcal{O}_{\mathbb{P}(A)}(-1)\longrightarrow A\otimes\mathcal{O}\xrightarrow{B}V\otimes\mathcal{O}\longrightarrow V/\mathcal{U}_3,$$

where $B\colon A\to V$ is general. Its vanishing requires $B(y)\in U_3$. The induced map $A\to V/W$ is an isomorphism, so the $\mathbb{P}(A)$ factor is the graph of an isomorphism with $\{U_3:W\subset U_3\}=\mathbb{P}(V/W)\simeq\mathbb{P}^3$. In particular π_2 and π_3 are isomorphisms onto their images. Thus the intermediate fivefold is

$$M=\{([u],L)\in\mathbb{P}(V)\times\mathbb{P}(V/W):u\bmod W\in L\}.$$

It is the zero locus of the universal quotient-bundle section on $\mathbb{P}(V/W)\times\mathbb{P}(V)$; the standard Cayley trick [1, §2, Cayley tricks] identifies $M\simeq\mathrm{Bl}_\ell\mathbb{P}(V)$, where $\ell=\mathbb{P}(W)$. Denote by E_M the exceptional divisor. The two natural classes satisfy $b=H-E_M$.

Description of π_{23}. The remaining equation is a section of

$$\det\mathcal{U}_3^\vee\otimes\mathcal{U}_1^\vee=\mathcal{O}_M(H+b)=\mathcal{O}_M(2H-E_M).$$

It comes from a tensor $T\in\mathbb{S}_{2,1,1}V^\vee\subset\bigwedge^3V^\vee\otimes V^\vee$. Off E_M, where $U_3=W+\langle u\rangle$, its equation is

$$q_T(u)=T(v_1\wedge v_2\wedge u,u)=0,$$

a quadric containing ℓ. The induced map

$$\mathbb{S}_{2,1,1}V^\vee\longrightarrow H^0(\mathbb{P}(V),\mathcal{I}_\ell(2)),\qquad T\longmapsto q_T$$

is surjective. Indeed, $\bigwedge^3V^\vee\otimes V^\vee$ splits into $\mathbb{S}_{2,1,1}V^\vee\oplus\bigwedge^4V^\vee$, and the alternating summand contributes zero to q_T. Given $p\in W^\perp$ and $r\in V^\vee$, choose $\lambda_1,\lambda_2\in V^\vee$ dual to v_1,v_2; then $(\lambda_1\wedge\lambda_2\wedge p)\otimes r$ induces $p(u)r(u)$. Such products span $H^0(\mathcal{I}_\ell(2))$. Hence q_T is a general quadric containing ℓ, smooth off ℓ by Bertini and smooth on ℓ because its four general normal linear derivatives have no common zero.

For $u\in W$ and $U_3=W+\langle z\rangle$, the equation on E_M is

$$T(v_1\wedge v_2\wedge z,u)=(dq_T)_u(z)=0.$$

It cuts the projectivized normal directions of the quadric along ℓ. Therefore the divisor $X \subset M$ is the strict transform of $Q^4 = Z(q_T)$, scheme-theoretically, and

$$X \simeq \mathrm{Bl}_\ell Q^4, \qquad E = E_M|_X = H - b, \qquad -K_X = 4H - 2E = 2H + 2b.$$

The normal sequence of $\ell \subset Q^4 \subset \mathbb{P}^5$ gives $N_{\ell/Q^4} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$, so $E \to \ell$ is a $\mathbb{P}^2$-bundle. This proves that π_{23} is an extremal divisorial contraction.

Description of π_1. The incidence above also identifies

$$M \simeq \mathbb{P}_{\mathbb{P}^3}\big(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)\big),$$

with relative hyperplane class H and base class b. For its projection $p\colon M \to \mathbb{P}^3$, the projective-bundle formula gives

$$p_*\mathcal{O}_M(H+b) = \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(2), \qquad R^i p_*\mathcal{O}_M(H+b) = 0 \quad (i > 0).$$

The induced section is (L_1, L_2, Q), with L_1, L_2 linear and Q quadratic on $\mathbb{P}^3$: in coordinates adapted to W the quadric is $q_T(s,t,z) = sL_1(z) + tL_2(z) + Q(z)$. Consequently X cuts a hyperplane in each fibre $\mathbb{P}^2$ of p, unless all three coefficients vanish, in which case the whole $\mathbb{P}^2$ remains. The jumping locus is the regular zero scheme

$$D = Z(L_1, L_2, Q) \subset \mathbb{P}^3, \qquad [D] = c_3\big(\mathcal{O}(1)^{\oplus 2} \oplus \mathcal{O}(2)\big) = 2b^3.$$

Since the quadratic system is general, $L_1 = L_2 = 0$ is a line on which Q has two distinct zeros. Thus there are exactly two special fibres, both $\mathbb{P}^2$, whereas all other fibres are $\mathbb{P}^1$. At either exceptional point the section is transverse, and the normal sequence along the special fibre F is

$$0 \longrightarrow N_{F/X} \longrightarrow \mathcal{O}_F^{\oplus 3} \longrightarrow \mathcal{O}_F(1) \longrightarrow 0, \qquad N_{F/X} \simeq \Omega^1_{\mathbb{P}^2}(1).$$

The fibres are connected and $\rho(X) = 2 = \rho(\mathbb{P}^3) + 1$; moreover $-K_X \cdot f = 2$. Hence π_1 is an elementary K-negative contraction of fibre type.

Nef and Mori cones. Let e be a line in a fibre of $E \to \ell$ and f a general $\mathbb{P}^1$-fibre of π_1. The two nef classes H, b, pulled back from Q^4 and $\mathbb{P}^3$ respectively, satisfy

	e	f
H	0	1
b	1	0
$E = H - b$	-1	1
$-K_X = 2H + 2b$	2	2

Since $\rho(X) = 2$, the two effective classes e, f are dual to the independent nef classes b, H. Thus $\mathrm{Nef}(X) = \langle H, b\rangle$ and $\overline{\mathrm{NE}}(X) = \langle [e], [f]\rangle$ are the full simplicial cones. The blow-up description also shows that X is rational.

◇

References

[1] M. Bernardara, E. Fatighenti, L. Manivel and F. Tanturri, *Fano fourfolds of K3 type*, Ann. Inst. Fourier (online 2026), doi:10.5802/aif.3761.

[2] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), 133–141.