Fano 2-431-90-a. $\mathscr{Z}\big(\mathrm{Fl}(1,2,6),\ \bigwedge^3(V_6/\mathcal{U}_2)^\vee \otimes \det \mathcal{U}_2^\vee \oplus (\mathcal{U}_2/\mathcal{U}_1)^\vee \otimes (\mathcal{U}_1^\vee)^2\big)$.

Invariants. $h^0(-K) = 90, \quad (-K)^4 = 431, \quad h^{1,1} = 2, \quad h^{2,2} = 3, \quad -\chi(T) = -11.$

Description. Regard $\mathrm{Fl}(1,2,6)$ as the incidence subvariety of $\mathbb{P}^5 \times \mathrm{Gr}(2,6)$, and let π_i forget the i-th factor. Set $H = c_1(\mathcal{U}_1^\vee|_X)$ and $h = c_1(\det \mathcal{U}_2^\vee|_X)$.

(1) $\pi_1 : X \simeq \mathrm{Bl}_{Q^2}\mathbb{P}^4 \to \mathbb{P}^4$ (1-625-126-a) is the K-negative extremal blow-up of a smooth quadric surface.

(2) $\pi_2 : X \simeq \mathrm{Bl}_p Q^4 \to Q^4$ (1-512-105-a) is the K-negative extremal blow-up of a point on a smooth quadric fourfold.

- $\mathrm{Nef}(X) = \langle H, h\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [2] gives $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(Q^2), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^4)\rangle$ and $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(p), \mathrm{D}^{\mathrm{b}}(p), \mathrm{D}^{\mathrm{b}}(p), \mathrm{D}^{\mathrm{b}}(Q^4)\rangle$, with the standard fully faithful embeddings and twists. By Kapranov's collection on Q^4 [3], $\mathrm{D}^{\mathrm{b}}(X)$ has a full exceptional collection of length nine.

Explanation. Fix a nonzero vector $v \in V_6$. With a choice of volume form, the first summand of the defining bundle is $(V_6/\mathcal{U}_2)$; hence its section imposes $v \in U_2$. Put $p = [v] \in \mathbb{P}(V_6)$. The resulting fivefold is

$$M = \{U_1 \subset U_2 :\ v \in U_2\} \simeq \mathbb{P}_{\mathbb{P}^4}(\mathcal{O} \oplus \mathcal{O}(-1)) \simeq \mathrm{Bl}_p \mathbb{P}^5,$$

where $\mathbb{P}^4 = \mathbb{P}(V_6/\langle v\rangle)$ and the exceptional divisor of $M \to \mathbb{P}^5$ has class $H - h$. The remaining summand is the line bundle

$$(\mathcal{U}_2/\mathcal{U}_1)^\vee \otimes (\mathcal{U}_1^\vee)^2 \simeq \det \mathcal{U}_2^\vee \otimes \mathcal{U}_1^\vee = \mathcal{O}_M(H + h).$$

Description of π_1. For the projection $q : M \to \mathbb{P}^4$, the projective bundle formula yields

$$q_*\mathcal{O}_M(H + h) \simeq \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(2), \qquad R^i q_*\mathcal{O}_M(H + h) = 0 \quad (i > 0).$$

More precisely, twisting the defining rank-two bundle by $\mathcal{O}(-1)$ identifies

$$M \simeq \mathbb{P}_{\mathbb{P}^4}\big(\mathcal{O}(-1) \oplus \mathcal{O}(-2)\big) = \mathbb{P}_{\mathbb{P}^4}\big((\mathcal{O}(1) \oplus \mathcal{O}(2))^\vee\big),$$

with $\mathcal{O}_M(H + h)$ as the relative ample line bundle. Thus the second defining equation corresponds to a section (ℓ, q_2) of $\mathcal{O}(1) \oplus \mathcal{O}(2)$ on $\mathbb{P}^4$. For general data, its zero locus

$$C = Z_{\mathbb{P}^4}(\ell, q_2) \simeq Q^2 \subset \mathbb{P}^3 \subset \mathbb{P}^4$$

is a smooth quadric surface of codimension two. All the hypotheses of the Cayley blow-up lemma [1, Lemma 3.2] are therefore satisfied, and $X \simeq \mathrm{Bl}_C \mathbb{P}^4$. The exceptional divisor of π_1 is $E' = 2h - H$; its fibres over C are $\mathbb{P}^1$.

Description of π_2. Under $M \simeq \mathrm{Bl}_p \mathbb{P}^5$, the remaining equation is a section of

$$\mathcal{O}_M(H + h) = \mathcal{O}_M(2H - E_M), \qquad E_M = H - h.$$

For the blow-up morphism $b : M \to \mathbb{P}^5$,

$$b_*\mathcal{O}_M(2H - E_M) = \mathcal{I}_p(2), \qquad R^i b_*\mathcal{O}_M(2H - E_M) = 0 \quad (i > 0).$$

Indeed, writing the original section as a tensor $T \in \mathbb{S}_{2,1}V_6^\vee$, it induces the quadratic form $q(u) = T(u, u, v)$, with $q(v) = 0$. These forms fill $H^0(\mathbb{P}^5, \mathcal{I}_p(2))$: the components $T(u, u, v)$ may be prescribed arbitrarily subject only to vanishing at v. Consequently the image is a general smooth quadric $Q^4 = Z(q) \subset \mathbb{P}^5$ through p. Since q has multiplicity one at p, its zero locus in M is the strict transform of Q^4; therefore

$$X \simeq \mathrm{Bl}_p Q^4, \qquad E = (H - h)|_X \simeq \mathbb{P}^3.$$

The map $\pi_1 \circ \pi_2^{-1}$ is the linear projection of Q^4 from p. Under this projection, the tangent hyperplane section $Q^4 \cap T_p Q^4$ is the cone over the quadric surface C of lines through p; its strict transform is $E' = 2h - H$. Conversely, the exceptional $\mathbb{P}^3$ of π_2 maps isomorphically to the hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ containing C.

The blow-up formula gives $-K_X = 4H - 3E = H + 3h = 5h - E'$, and yields $(-K_X)^4 = 2 \cdot 4^4 - 3^4 = 431$. Moreover, quartics on Q^4 vanishing to order three at p give $h^0(-K_X) = 105 - 15 = 90$. Let e be a line in E and f a fibre of $E' \to C$. Then

$$(H \cdot e, h \cdot e) = (0, 1), \qquad (H \cdot f, h \cdot f) = (1, 0).$$

Since H and h are nef pullbacks by the two contractions and $\rho(X) = 2$, they generate the full nef cone, and both contractions are K-negative. Finally X is rational because it is birational to Q^4. The blow-up description also gives $h^{2,2}(X) = 3$, while $\mathrm{Aut}(X)$ is the stabilizer of p in $\mathrm{Aut}(Q^4)$, of dimension 11. ◇

References

[1] M. Bernardara, E. Fatighenti, L. Manivel and F. Tanturri, *Fano fourfolds of K3 type*, Ann. Inst. Fourier (Grenoble) (online 2026), doi:10.5802/aif.3761.

[2] D. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), 133–141.

[3] M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), 479–508.