**Fano 2-432-90-a.**
$\mathscr{Z}\big(\mathrm{Fl}(1,2,4),\ \mathcal{U}_1^\vee\big)$, with $\mathrm{Fl}(1,2,4) \subset \mathbb{P}^3 \times \mathrm{Gr}(2,4)$.

*Invariants.* $h^0(-K) = 90,\ (-K)^4 = 432,\ h^{1,1} = 2,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 3,\ -\chi(T) = -12$.

*Description.*

(1) $\pi_1 \colon X \to \mathrm{Gr}(2,4)$ is an extremal divisorial contraction, the blow-up of the quadric fourfold $\mathrm{Gr}(2,4)$ (1-512-105-a) along a plane.
(2) $\pi_2 \colon X \to \mathbb{P}^3$ is an extremal contraction of fiber type onto a plane, the $\mathbb{P}^2$-bundle $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O})$; it resolves the linear projection of $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ from the plane.

- $a, L$ denote the pullbacks of the hyperplane class of $\mathbb{P}^3$ and of the Plücker class.
- $\mathrm{Nef}(X) = \langle a, L\rangle$ is simplicial.
- $\mathrm{Aut}(X)$ has dimension 12.
- $X$ is rational.

*Semiorthogonal decompositions.*

- From $\pi_1$, [Orl] and [Kap]: $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2), \pi_1^* \mathrm{D}^{\mathrm{b}}(\mathrm{Gr}(2,4))\rangle$, a full exceptional collection of length $3 + 6 = 9$.

*Explanation.*

Write $\mathrm{Fl}(1,2,4) = \{u \in U_2 \subset V_4\}$. The section of $\mathcal{U}_1^\vee$ is a linear form on $V_4$, with kernel a hyperplane $V_3$, and imposes $u \in V_3$. Thus

$$X = \{(u, U_2) :\ u \in U_2 \cap V_3\}.$$

**Description of $\pi_1$.** The projection $\mathrm{Fl}(1,2,4) \to \mathrm{Gr}(2,4)$ is $\mathbb{P}(\mathcal{U}_2)$, and the section of $\mathcal{U}_1^\vee = \mathcal{O}_{\mathbb{P}(\mathcal{U}_2)}(1)$ is induced by the section of $\mathcal{U}_2^\vee$ given by the same linear form. Its zero locus is $\Pi = \mathrm{Gr}(2, V_3)$, a plane of the quadric $\mathrm{Gr}(2,4)$, smooth of codimension 2; so [BFMT, Lemma 3.2] gives $X \simeq \mathrm{Bl}_\Pi \mathrm{Gr}(2,4)$. The fiber over $U_2$ is the point $U_2 \cap V_3$ if $U_2 \not\subset V_3$, and the line $\mathbb{P}(U_2)$ if $U_2 \in \Pi$. Such a line $f$ has $a \cdot f = 1$ and $L \cdot f = 0$.
**Description of $\pi_2$.** The image of $\pi_2$ is the plane $\mathbb{P}(V_3)$, and the fiber over $u$ is the plane $\mathbb{P}(V_4/u)$ of lines of $\mathbb{P}^3$ through $u$. So $X = \mathbb{P}_{\mathbb{P}^2}(V_4 \otimes \mathcal{O}/\mathcal{O}(-1))$, and the sequence

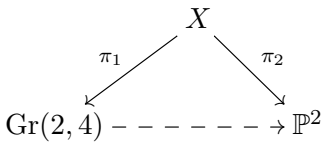
$$0 \to V_3 \otimes \mathcal{O}/\mathcal{O}(-1) \to V_4 \otimes \mathcal{O}/\mathcal{O}(-1) \to V_4/V_3 \otimes \mathcal{O} \to 0, \qquad V_3 \otimes \mathcal{O}/\mathcal{O}(-1) \simeq T_{\mathbb{P}^2}(-1),$$

splits since $H^1(\mathbb{P}^2, T_{\mathbb{P}^2}(-1)) = 0$; hence $X \simeq \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}(-1) \oplus \mathcal{O})$. A line $g$ in a fiber has $L \cdot g = 1$ and $a \cdot g = 0$. Off $\Pi$, the composition $\pi_2 \circ \pi_1^{-1}$ is $U_2 \mapsto U_2 \cap V_3$; it is given by the linear forms on $\bigwedge^2 V_4$ vanishing on $\bigwedge^2 V_3$, that is the linear projection of $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ from the plane $\Pi$.

**Cones.** The classes $a, L$ are nef, and $f, g$ form the dual basis:

| | $f$ | $g$ |
|---|---|---|
| $a$ | 1 | 0 |
| $L$ | 0 | 1 |

Hence $\mathrm{Nef}(X) = \langle a, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f, g\rangle$ are dual simplicial cones, and $\pi_1, \pi_2$ are the two extremal contractions.




**Hodge numbers and automorphisms.** From $\pi_1$, $h^{2,2}(X) = 2+1 = 3$, and $X$ is rational. Every automorphism of $X$ preserves both contractions, hence the pair $(\mathrm{Gr}(2,4), \Pi)$, and conversely every automorphism of $\mathrm{Gr}(2,4)$ preserving $\Pi$ lifts to $X$. These are induced by the elements of $\mathrm{PGL}(V_4)$ preserving $V_3$, a group of dimension $15-3 = 12$. The $(1,1)$-divisor of $\mathbb{P}^2 \times \mathbb{P}^3$ (2-432-90-d) has the same invariants, but both its contractions are of fiber type, while $\pi_1$ is birational.

◇

## References

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