Fano 2-432-90-d. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^3,\ \mathcal{O}(1,1)\big)$.

Invariants. $h^0(-K) = 90,\ (-K)^4 = 432,\ h^{1,1} = 2,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 3,\ -\chi(T) = -12$.

Description.

(1) $\pi_1 \colon X \to \mathbb{P}^3$ is an extremal contraction of fiber type, whose fibers are lines, except for one fiber, which is a plane.
(2) $\pi_2 \colon X \to \mathbb{P}^2$ is an extremal contraction of fiber type, the $\mathbb{P}^2$-bundle $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \Omega_{\mathbb{P}^2}(1))$.

- h, a denote the pullbacks of the hyperplane classes of $\mathbb{P}^2$ and $\mathbb{P}^3$.
- $\mathrm{Nef}(X) = \langle h, a \rangle$ is simplicial.
- $\mathrm{Aut}(X)$ has dimension 12.
- X is rational.

Semiorthogonal decompositions.

- From π_2 [Orl]: $\mathrm{D}^{\mathrm{b}}(X) = \langle \pi_2^* \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2), \pi_2^* \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2)(a), \pi_2^* \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2)(2a) \rangle$, a full exceptional collection of length 9.

Explanation.

Write $\mathbb{P}^2 = \mathbb{P}(V_3)$ and $\mathbb{P}^3 = \mathbb{P}(V_4)$. The section of $\mathcal{O}(1,1)$ is a general bilinear form B on $V_3 \times V_4$, that is a linear map $V_4 \to V_3^\vee$ of rank 3, with kernel a point $x_0 \in \mathbb{P}^3$. Thus $X = \{(z, x) : B(z, x) = 0\}$; it is smooth, since the matrix has maximal rank.

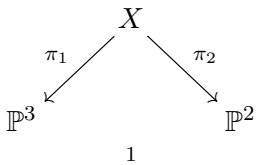
Description of π_1. The fiber of π_1 over x is $\{z : B(z, x) = 0\}$: a line of $\mathbb{P}^2$ if $x \neq x_0$, and the whole plane $\mathbb{P}^2$ if $x = x_0$. A line ℓ_h in a fiber has $h \cdot \ell_h = 1$ and $a \cdot \ell_h = 0$, and the nef class a vanishes exactly on the curves of the fibers.

Description of π_2. The linear form $B(z, \cdot)$ on V_4 never vanishes, so its kernel is a rank-3 subbundle $\mathcal{N} = \ker(V_4 \otimes \mathcal{O} \to \mathcal{O}_{\mathbb{P}^2}(1))$, and $X = \mathbb{P}_{\mathbb{P}^2}(\mathcal{N})$. The bundle $\mathcal{N}$ contains $x_0 \otimes \mathcal{O}$, and $\mathcal{N}/x_0 \simeq \ker(V_3^\vee \otimes \mathcal{O} \to \mathcal{O}(1)) = \Omega_{\mathbb{P}^2}(1)$ by the Euler sequence; the extension splits since $H^1(\mathbb{P}^2, T_{\mathbb{P}^2}(-1)) = 0$, so $\mathcal{N} \simeq \mathcal{O} \oplus \Omega_{\mathbb{P}^2}(1)$. The section given by the summand $\mathcal{O}$ is the plane $\mathbb{P}^2 \times \{x_0\}$, the special fiber of π_1. A line ℓ_a in a fiber has $a \cdot \ell_a = 1$ and $h \cdot \ell_a = 0$, and the nef class h vanishes exactly on the curves of the fibers.

Cones and automorphisms. The classes h, a are nef, and ℓ_h, ℓ_a form the dual basis:

	ℓ_h	ℓ_a
h	1	0
a	0	1

Hence $\mathrm{Nef}(X) = \langle h, a \rangle$ and $\overline{\mathrm{NE}}(X) = \langle \ell_h, \ell_a \rangle$ are dual simplicial cones, and π_1, π_2 are the two extremal contractions.



Since π_1 is not equidimensional and π_2 is, every automorphism of X preserves both contractions. So $\mathrm{Aut}(X)$ is the stabilizer of B in $(\mathrm{GL}(V_3) \times \mathrm{GL}(V_4))/\mathbb{C}^*$; as the matrices of rank 3 form an open orbit, it has dimension $9+16-1-12 = 12$. Finally, X is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$, hence rational.

$\diamond$

References

[Orl] D. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. 41 (1993), 133–141.