Fano 2-433-90-a. $X = \mathscr{Z}(\mathrm{Fl}(1,3,5),\ (\mathcal{U}_1^\vee \otimes (\mathcal{U}_3/\mathcal{U}_1)^\vee)^{\oplus 2})$.

Invariants. $h^0(-K_X) = 90, \quad (-K_X)^4 = 433, \quad h^{1,1} = 2, \quad h^{2,2} = 3, \quad -\chi(T_X) = -13.$

Description. Regard $\mathrm{Fl}(1,3,5)$ as the incidence in $\mathbb{P}^4 \times \mathrm{Gr}(3,5)$, and let π_i omit the i-th factor. Set $H = c_1(\mathcal{U}_1^\vee|_X)$ and $b = c_1(\det \mathcal{U}_3^\vee|_X)$.

(1) $\pi_1 \colon X \to Y \subset \mathrm{Gr}(3,5)$ is a K-negative extremal small contraction onto a fourfold of Plücker degree 8 with one singular point, contracting a plane $\mathbb{P}^2$.
(2) $\pi_2 \colon X \simeq \mathrm{Bl}_C \mathbb{P}^4 \to \mathbb{P}^4$ is a K-negative extremal divisorial contraction, with centre a smooth plane conic C.

- $\mathrm{Nef}(X) = \langle H, b\rangle$, whose extremal rays induce π_2 and π_1, respectively.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [Orl] gives, up to the standard embeddings and twists,

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(C), \mathrm{D}^{\mathrm{b}}(C), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^4)\rangle.$$

Since $C \simeq \mathbb{P}^1$, X admits a full exceptional collection of length 9.

Explanation. Let V_5 be a five-dimensional vector space. The two sections correspond to a general pencil $\omega_{[s:t]} = s\omega_0 + t\omega_1$ of alternating forms on V_5, and X parametrizes flags $u \subset U_3 \subset V_5$ such that $\omega_0(u, U_3) = \omega_1(u, U_3) = 0$. A general pencil can be written, for a basis e_0, e_1, e_2, f_0, f_1 of V_5, as

$$\omega_0 = e_0^\vee \wedge f_0^\vee + e_1^\vee \wedge f_1^\vee, \qquad \omega_1 = e_1^\vee \wedge f_0^\vee + e_2^\vee \wedge f_1^\vee.$$

Every member has rank four, with kernel generated by $t^2 e_0 - ste_1 + s^2 e_2$. Thus the kernel lines form a smooth conic $C \subset \Pi_0 = \mathbb{P}(U_0) \subset \mathbb{P}(V_5)$, where $U_0 = \langle e_0, e_1, e_2\rangle$. Adjunction gives $-K_X = H + 2b$.

Description of π_1. Over $G = \mathrm{Gr}(3, V_5)$ the flag is $\mathbb{P}_G(\mathcal{U}_3)$, with projection p. The relative Euler sequence gives

$$p_*(\mathcal{U}_1^\vee \otimes (\mathcal{U}_3/\mathcal{U}_1)^\vee) = \bigwedge^2 \mathcal{U}_3^\vee.$$

All higher direct images of this summand vanish. Consequently the pencil induces

$$\varphi \colon \mathcal{O}_G^{\oplus 2} \longrightarrow \bigwedge^2 \mathcal{U}_3^\vee, \qquad (s,t) \longmapsto s\omega_0|_{\mathcal{U}_3} + t\omega_1|_{\mathcal{U}_3}.$$

A point U_3 lies in the image precisely when the two restrictions have a common kernel line, or equivalently when they are dependent in $\bigwedge^2 U_3^\vee$. Thus $Y = D_1(\varphi)$ scheme-theoretically; it has expected codimension two and class

$$[Y] = c_2(\bigwedge^2 \mathcal{U}_3^\vee), \qquad \int_G [Y]\, b^4 = 8.$$

The rank-zero locus consists of $[U_0]$. Indeed, a three-space isotropic for a rank-four alternating form on V_5 contains its kernel, so a common isotropic three-space contains all the kernel lines, which span U_0. In the affine chart of G parametrizing graphs $U_0 \to \langle f_0, f_1\rangle$, the six components of φ have independent linear terms at U_0. The local equations of Y are therefore the 2×2 minors of a 2×3 matrix of regular parameters. Hence Y has a single isolated determinantal singularity at $[U_0]$ and is smooth elsewhere.

Above $Y \setminus \{[U_0]\}$ the common kernel is a single line, while above $[U_0]$ the fibre is $\widetilde{\Pi}_0 = \mathbb{P}(U_0) \simeq \mathbb{P}^2$. As π_2 identifies $\widetilde{\Pi}_0$ with the strict transform of Π_0, the blow-up description below gives

$$N_{\widetilde{\Pi}_0/X} \simeq N_{\Pi_0/\mathbb{P}^4} \otimes \mathcal{O}_{\Pi_0}(-C) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}.$$

Thus π_1 is small, and $(-K_X)\cdot \ell = 1$ for a line $\ell \subset \widetilde{\Pi}_0$.

Description of π_2. Over $\mathbb{P}(V_5)$, the flag is $\mathrm{Gr}(2, V_5/\mathcal{U}_1)$. The relative pushforward of each defining summand is

$$(V_5/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee \simeq \Omega^1_{\mathbb{P}^4}(2).$$

All its higher direct images vanish. Thus the two sections give a bundle morphism

$$\psi\colon \mathcal{O}_{\mathbb{P}^4}^{\oplus 2} \longrightarrow \Omega^1_{\mathbb{P}^4}(2), \qquad \psi_u = (\omega_0(u,-), \omega_1(u,-)).$$

Its rank-one degeneracy locus is precisely the conic C of kernel lines; the rank-zero locus is empty. The expected codimension is three, and Thom–Porteous gives

$$[C] = c_3(\Omega^1_{\mathbb{P}^4}(2)) = \left[\frac{(1+H)^5}{1+2H}\right]_3 = 2H^3.$$

For $u \notin C$, the annihilator of $\operatorname{Im}\psi_u$ has dimension two and fixes U_3/u. For $u \in C$, it has dimension three, giving a fibre $\mathrm{Gr}(2,3) \simeq \mathbb{P}^2$. Since C is smooth of the expected codimension, locally along C one can reduce ψ to a nonzero constant entry and three generators of $\mathcal{I}_C$. The resulting incidence is the standard local blow-up model for this ideal. Hence $X \simeq \mathrm{Bl}_C\,\mathbb{P}^4$ scheme-theoretically. The exceptional divisor is a $\mathbb{P}^2$-bundle over C; from $-K_X = 5H - 2E = H + 2b$ we obtain $E = 2H - b$. In particular, $b = 2H - E$ is the linear system of quadrics through C, and the induced map $\mathbb{P}^4 \dashrightarrow Y \subset \mathbb{P}^9$ contracts the plane Π_0 to $[U_0]$.

Mori geometry. Let e be a line in a fibre of $E \to C$, and $\ell \subset \widetilde{\Pi}_0$ a line. Then

$$(H \cdot e, b \cdot e) = (0,1), \qquad (H \cdot \ell, b \cdot \ell) = (1,0).$$

Since $\rho(X) = 2$ and H, b are nef, they generate the nef cone and support exactly the two K-negative extremal contractions π_2 and π_1. Finally, X is rational as a blow-up of $\mathbb{P}^4$. ◇

References

[Orl] D. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.