Fano 2-480-99-a. $\mathscr{Z}\big(\mathrm{Fl}(1,2,6)\times\mathbb{P}^4,\ \mathcal{Q}\oplus\mathcal{Q}\boxtimes\mathcal{O}_{\mathbb{P}^4}(1)\oplus\det\mathcal{U}_2^{\vee\otimes 2}\big)$, with $\mathcal{Q}=\bigwedge^3(V_6/\mathcal{U}_2)^\vee\otimes\det\mathcal{U}_2^\vee\simeq V_6/\mathcal{U}_2$.

Invariants. $h^0(-K)=99,\ (-K)^4=480,\ h^{1,1}=2,\ h^{2,1}=0,\ h^{3,1}=0,\ h^{2,2}=2,\ -\chi(T)=-16$.

Description. Let $Q^3\subset\mathbb{P}^4$ be a smooth quadric threefold, and let $C\subset\mathbb{P}^5$ be its projective cone, with vertex p. The two extremal contractions of X are

(1) The $\mathbb{P}^1$-bundle
$$\pi_1\colon X\simeq\mathbb{P}_{Q^3}(\mathcal{O}\oplus\mathcal{O}(-1))\longrightarrow Q^3.$$
(2) The divisorial contraction
$$\pi_{23}\colon X\simeq\mathrm{Bl}_p(C)\longrightarrow C,$$
the projection to $\mathbb{P}^5$ forgetting the second and third factors of $\mathbb{P}^5\times\mathrm{Gr}(2,6)\times\mathbb{P}^4$, which contracts the section $E_0\simeq Q^3$, with $N_{E_0/X}=\mathcal{O}(-1)$, to the vertex p.

The projections to $\mathbb{P}^5\times\mathbb{P}^4$ and $\mathbb{P}^5\times\mathrm{Gr}(2,6)$ are isomorphisms onto their images. Moreover, X is rational and has index 2 and $\dim\mathrm{Aut}(X)=16$.

Semiorthogonal decompositions. From π_1, Orlov's projective bundle formula [2] gives
$$\mathrm{D^b}(X)=\langle\pi_1^*\,\mathrm{D^b}(Q^3),\ \pi_1^*\,\mathrm{D^b}(Q^3)(H)\rangle.$$
Since Q^3 admits a full exceptional collection of length 4 by Kapranov [1], this yields a full exceptional collection on X of length $8=\chi_{\mathrm{top}}(X)$.

Explanation of the description.

Setup. We regard $\mathrm{Fl}(1,2,V_6)$ as the incidence variety of pairs $U_1\subset U_2\subset V_6$, with $\dim U_i=i$, in $\mathbb{P}(V_6)\times\mathrm{Gr}(2,V_6)$, and work on its product with $\mathbb{P}(V_5)$. The tautological bundles are denoted by $\mathcal{U}_1\subset\mathcal{U}_2$. On $\mathrm{Gr}(2,V_6)$, put $\mathcal{Q}'=V_6/\mathcal{U}_2$, of rank 4. Using a volume form on V_6, we obtain $\det\mathcal{Q}'\simeq\det\mathcal{U}_2^\vee$, and so
$$\bigwedge^3\mathcal{Q}'^\vee\otimes\det\mathcal{U}_2^\vee\simeq\mathcal{Q}'\otimes\det\mathcal{Q}'^\vee\otimes\det\mathcal{U}_2^\vee=\mathcal{Q}'.$$
Moreover we have $(\mathcal{U}_2/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee=\det\mathcal{U}_2^\vee$, hence the first two summands in the vector bundle defininng X are $\mathcal{Q}'$ and $\mathcal{Q}'\boxtimes\mathcal{O}(1)$, and the third is $\mathcal{O}_{\mathrm{Gr}}(2)$. Their sections are as described: v vanishes iff $v\in U_2$; A vanishes at (U_2,y) iff the composite
$$\mathcal{O}(0,-1)\xrightarrow{\ y\ }V_5\otimes\mathcal{O}\xrightarrow{\ A\ }V_6\otimes\mathcal{O}\longrightarrow\mathcal{Q}'$$
vanishes, i.e. $A(y)\in U_2$.

Thus, if $q\in H^0(\mathrm{Gr}(2,V_6),\mathcal{O}(2))$ denotes the third section, the zero locus is
$$X=\{(U_1,U_2,y):U_1\subset U_2,\ v\in U_2,\ A(y)\in U_2,\ q(U_2)=0\}.$$

Now, the condition $v \in U_2$ means that U_2 is a two-dimensional subspace of V_6 containing the fixed line $\langle v \rangle$. Such subspaces are parametrized by

$$\{U_2 \in \mathrm{Gr}(2, V_6) : v \in U_2\} \simeq \mathbb{P}(V_6/\langle v \rangle) \simeq \mathbb{P}^4,$$

where U_2 corresponds to the line $U_2/\langle v \rangle$ in $V_6/\langle v \rangle$. For general (A, v), the induced linear map

$$\bar{A} : V_5 \longrightarrow V_6/\langle v \rangle$$

is an isomorphism. Writing $y = [w] \in \mathbb{P}(V_5)$, the condition $A(w) \in U_2$ is equivalent to

$$\bar{A}(w) \in U_2/\langle v \rangle.$$

Since $\bar{A}$ is an isomorphism, this determines the line $\langle w \rangle$ uniquely:

$$\langle w \rangle = \bar{A}^{-1}(U_2/\langle v \rangle).$$

Thus U_2 uniquely determines the point $y \in \mathbb{P}(V_5)$. Conversely, $y = [w]$ determines $U_2 = \langle v, A(w) \rangle$, since $A(w) \notin \langle v \rangle$.

Consequently, the incidence condition $A(y) \in U_2$, together with $v \in U_2$, identifies the $\mathbb{P}^4$ factor with the space of two-planes containing v. In particular, the projections π_2 and π_3 are isomorphisms onto their images.

The remaining equation $q(U_2) = 0$ cuts out a quadric threefold

$$Q^3 = \{U_2 \ni v : q(U_2) = 0\} \subset \mathbb{P}(V_6/\langle v \rangle).$$

Since the restriction of sections of $\mathcal{O}_{\mathrm{Gr}(2,6)}(2)$ to this linearly embedded $\mathbb{P}^4$ is surjective, the variety Q^3 is a general, hence smooth, quadric threefold. Thus the image of π_1 is the graph of the correspondence $U_2 \leftrightarrow y$, and is isomorphic to Q^3.

Description of π_1. The projection $\mathrm{Fl}(1, 2, 6) \to \mathrm{Gr}(2, 6)$ is $\mathbb{P}(\mathcal{U}_2) \to \mathrm{Gr}(2, 6)$. On Q^3,

$$0 \to \mathcal{O} \xrightarrow{v} \mathcal{U}_2 \to \mathcal{O}_{Q^3}(-1) \to 0,$$

where the quotient is the tautological line of $\mathbb{P}(V_6/\langle v \rangle)$. This sequence splits because $H^1(Q^3, \mathcal{O}(1)) = 0$. Hence $X = \mathbb{P}_{Q^3}(\mathcal{O} \oplus \mathcal{O}(-1))$, and $h = c_1(\mathcal{U}_2^\vee)$ is the hyperplane class of Q^3. The fiber $\mathbb{P}(U_2)$ is a line of $\mathbb{P}^5$ through p.

Description of π_{23}. Write $H = c_1(\mathcal{U}_1^\vee)$ and $h = c_1(\mathcal{U}_2^\vee)$ on X. The image is the quadric cone

$$C = \{x \in \mathbb{P}(V_6) : \ \bar{q}(x \bmod v) = 0\},$$

where $Q^3 = \{\bar{q} = 0\} \subset \mathbb{P}(V_6/\langle v \rangle)$. It has corank one, with isolated vertex $p = [v]$, and is normal, Gorenstein and terminal, with $-K_C = \mathcal{O}_C(4)$.

The projection $(U_1, U_2, y) \mapsto U_1$ has image $\bigcup_{U_2 \in Q^3} \mathbb{P}(U_2) = C$. It is injective away from $U_1 = \langle v \rangle$, and it contracts the section $E_0 = \mathbb{P}(\mathcal{O}) \simeq Q^3$ to p. The normal bundle is

$$N_{E_0/X} = \mathcal{H}om\big(\mathcal{O}, \mathcal{O}(-1)\big) = \mathcal{O}(-1),$$

so π_{23} is the blow-up of the vertex of C. Since $H|_{E_0} = 0$ and $E_0|_{E_0} = -h$, we get $E_0 = H - h$. By adjunction, $-K_X = 2H + 2h = 4H - 2E_0$, so X has index 2. As a projective bundle over a smooth quadric, X is rational.

Extremality of π_{23}. The class $H = \pi_{23}^*\mathcal{O}_{\mathbb{P}^5}(1)$ is nef, and $H \cdot e = 0$ and $H \cdot f = 1$. Since $\rho(X) = 2$ and $\overline{\mathrm{NE}}(X) = \langle e, f\rangle$ (see the paragraph on the nef and Mori cones), the face $H^\perp \cap \overline{\mathrm{NE}}(X)$ is the ray $\mathbb{R}_{\geq 0}[e]$, and every curve with $H \cdot \Gamma = 0$ lies in E_0 because π_{23} is injective elsewhere. It follows that π_{23} is the contraction of $\mathbb{R}_{\geq 0}[e]$, which is clearly K-negative. Its exceptional locus is the divisor E_0, which gets contracted to a point.

Nef and Mori cones of X. The classes h and H are nef, being pulled back by π_1 and π_{23}, and

	e	f
h	1	0
H	0	1
$E_0 = H - h$	-1	1
$-K$	2	2

Hence $\mathrm{Nef}(X) = \langle h, H\rangle$ and $\overline{\mathrm{NE}}(X) = \langle e, f\rangle$. The two extremal contractions are π_1 and π_{23}.

$$\begin{array}{ccc} & X & \\ {\scriptstyle \pi_{23}}\swarrow & & \searrow{\scriptstyle \pi_1} \\ C & \dashrightarrow & Q^3 \end{array} .$$

Here the bottom arrow is the projection from the vertex

Automorphisms of X. Write $X = \mathbb{P}_{Q^3}(E)$, where $E = \mathcal{O}_{Q^3} \oplus \mathcal{O}_{Q^3}(-1)$, and denote the structural morphism $X \to Q^3$ by π. We have shown that X has exactly two extremal contractions, namely π, of fiber type, and π_{23}, which is divisorial. Since automorphisms preserve the Mori cone and the type of its extremal contractions, every automorphism of X preserves π and therefore induces an automorphism of Q^3.

Conversely, every automorphism of Q^3 lifts to X. Indeed, $\mathrm{Aut}(Q^3) \simeq \mathrm{SO}_5$ acts on the tautological line bundle $\mathcal{O}_{Q^3}(-1)$ through its standard action on $\mathbb{P}^4$, and hence acts on E and its projectivization. Thus the homomorphism

$$\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(Q^3)$$

is surjective, and its kernel is $\mathrm{Aut}_{Q^3}(X)$, the group of automorphisms acting trivially on the base.

We now compute this kernel. Every automorphism of $\mathbb{P}(E)$ over Q^3 is induced by an automorphism of E, unique up to multiplication by a nonzero scalar. To justify this, observe that such an automorphism g satisfies

$$g^*\mathcal{O}_{\mathbb{P}(E)}(1) \simeq \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*L$$

for some line bundle L on Q^3. Pushing forward gives $E^\vee \simeq E^\vee \otimes L$, hence $L^{\otimes 2} \simeq \mathcal{O}_{Q^3}$ by taking determinants. Since $\mathrm{Pic}(Q^3) \simeq \mathbb{Z}$, we obtain $L \simeq \mathcal{O}_{Q^3}$, proving the assertion.

Now

$$\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}) = H^0(Q^3, \mathcal{O}(1)), \qquad \mathrm{Hom}(\mathcal{O}, \mathcal{O}(-1)) = 0.$$

Consequently, every automorphism of E is represented by a matrix

$$\begin{pmatrix} a & s \\ 0 & d \end{pmatrix}, \qquad a, d \in \mathbb{C}^*, \quad s \in H^0(Q^3, \mathcal{O}(1)).$$

Since multiplication of the entire matrix by a nonzero scalar does not change its action on $\mathbb{P}(E)$, we may divide by d. Thus each element of $\mathrm{Aut}_{Q^3}(X)$ is represented uniquely by a matrix

$$g_{\lambda,s} = \begin{pmatrix} \lambda & s \\ 0 & 1 \end{pmatrix}, \qquad \lambda \in \mathbb{C}^*, \quad s \in H^0(Q^3, \mathcal{O}(1)).$$

Matrix multiplication gives

$$g_{\lambda,s} g_{\mu,t} = g_{\lambda\mu,\, s+\lambda t}.$$

It follows that

$$\mathrm{Aut}_{Q^3}(X) \simeq H^0(Q^3, \mathcal{O}(1)) \rtimes \mathbb{C}^* \simeq \mathbb{G}_a^5 \rtimes \mathbb{G}_m,$$

where $\mathbb{C}^*$ acts on $H^0(Q^3, \mathcal{O}(1))$ by scalar multiplication. In particular, the subgroup corresponding to $\lambda = 1$ is unipotent, since $(g_{1,s} - I)^2 = 0$. As $h^0(Q^3, \mathcal{O}(1)) = 5$, the kernel has dimension 6.

The action of SO_5 provides a splitting of the homomorphism $\mathrm{Aut}(X) \to \mathrm{Aut}(Q^3)$. Therefore

$$\mathrm{Aut}(X) \simeq (\mathbb{G}_a^5 \rtimes \mathbb{G}_m) \rtimes \mathrm{SO}_5,$$

and hence

$$\dim \mathrm{Aut}(X) = 5 + 1 + 10 = 16.$$

$\diamond$

References

[1] M. M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), 479–508.

[2] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), no. 4, 852–862.