Fano 2-512-105-b. $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^4,\ \mathcal{O}(1,1)\big)$.

Invariants. $h^0(-K) = 105$, $(-K)^4 = 512$, $h^{1,1} = 2$, $h^{2,1} = 0$, $h^{3,1} = 0$, $h^{2,2} = 2$, $-\chi(T) = -18$.

Description.

(1) The projection $\pi_1 \colon X \to \mathbb{P}^4$ is the blow-up of $\mathbb{P}^4$ (type 1-625-126-a) along a plane $\Pi \simeq \mathbb{P}^2$. It is an extremal divisorial contraction.
(2) The projection $\pi_2 \colon X \to \mathbb{P}^1$ is the $\mathbb{P}^3$-bundle $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}^{\oplus 3} \oplus \mathcal{O}(-1))$. It is an extremal contraction of fiber type.

In particular, X is rational.

Semiorthogonal decompositions. By Orlov's blow-up and projective-bundle formulas [Orl92], we have

- From π_1, with $j \colon E \hookrightarrow X$ and $q \colon E \to \Pi$,

$$\mathrm{D}^{\mathrm{b}}(X) = \langle j_*\big(q^*\,\mathrm{D}^{\mathrm{b}}(\Pi) \otimes \mathcal{O}_E(-1)\big),\ \pi_1^*\,\mathrm{D}^{\mathrm{b}}(\mathbb{P}^4)\rangle.$$

  By Beilinson's exceptional collections on projective spaces [Bei78], this gives a full exceptional collection of length $3 + 5 = 8 = \chi_{\mathrm{top}}(X)$.
- From π_2,

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \pi_2^*\,\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1),\ \pi_2^*\,\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)(H),\ \pi_2^*\,\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)(2H),\ \pi_2^*\,\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)(3H)\rangle.$$

Explanation.

Setup. The factors are ordered as $\mathbb{P}^1 \times \mathbb{P}^4$. A general section of $\mathcal{O}(1,1)$ is $t_0\lambda_0(y) + t_1\lambda_1(y)$, with $\lambda_0, \lambda_1 \in H^0(\mathbb{P}^4, \mathcal{O}(1))$ independent. Setting $\Pi = \{\lambda_0 = \lambda_1 = 0\} \simeq \mathbb{P}^2$, we have

$$X = \{(t, y) \in \mathbb{P}^1 \times \mathbb{P}^4 :\ t_0\lambda_0(y) + t_1\lambda_1(y) = 0\}.$$

Write $h = \mathcal{O}(1,0)|_X$ and $H = \mathcal{O}(0,1)|_X$. Adjunction gives $-K_X = (2h + 5H) - (h + H) = h + 4H$. Since $[X] = h + H$, the intersection numbers are

$$H^4 = 1, \qquad hH^3 = 1, \qquad h^2 = 0.$$

Hence $(h + 4H)^4 = 256 + 4 \cdot 64 = 512$, in agreement with FanoBase.

Description of π_1. Writing the section as $v_0 \otimes \lambda_0 + v_1 \otimes \lambda_1$, with $\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{Q}_{\mathbb{P}^1}$, this is [BFMT, §2, first Cayley lemma] with $m = 1$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^4}(1)$. Hence $X = \mathrm{Bl}_Z\,\mathbb{P}^4$ with $Z = \{\lambda_0 = \lambda_1 = 0\} = \Pi$. Off Π the point $t = [-\lambda_1(y) : \lambda_0(y)]$ is determined, and over Π the fiber is $\mathbb{P}^1$. So $E \simeq \Pi \times \mathbb{P}^1$, with $h = H - E$ and $-K_X = 5H - E$. As a check,

$$h^0(-K_X) = h^0(\mathcal{O}_{\mathbb{P}^4}(5)) - h^0(\mathcal{O}_\Pi(5)) = 126 - 21 = 105.$$

Description of π_2. For $p\colon \mathbb{P}^1 \times \mathbb{P}^4 \to \mathbb{P}^1$ we have $p_*\mathcal{O}(1,1) = V_5^\vee \otimes \mathcal{O}(1)$. The section is the composite

$$\mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{(t_0,t_1)} \mathcal{O}^{\oplus 2} \xrightarrow{(\lambda_0,\lambda_1)} V_5^\vee \otimes \mathcal{O},$$

a nowhere vanishing linear form λ_t on V_5. Hence $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{K})$, where $\mathcal{K} = \ker(V_5 \otimes \mathcal{O} \to \mathcal{O}(1))$. Since $\mathcal{K} \supset \ker\lambda_0 \cap \ker\lambda_1 = V_3 \otimes \mathcal{O}$ with quotient $\mathcal{O}(-1)$, we get $\mathcal{K} \simeq \mathcal{O}^{\oplus 3} \oplus \mathcal{O}(-1)$. The fibers are the hyperplanes $\{\lambda_t = 0\} \supset \Pi$.

Cones. The classes h and H are nef, being pulled back by π_2 and π_1, and

	e	f
h	1	0
H	0	1
$E = H - h$	-1	1
$-K$	1	4

Hence $\mathrm{Nef}(X) = \langle h, H\rangle$ and $\overline{\mathrm{NE}}(X) = \langle e, f\rangle$ are simplicial cones, and π_1, π_2 are the two extremal contractions. The blow-up description also shows that X is toric; its automorphism group is the stabilizer of Π in PGL_5, of dimension $18 = \chi(T_X)$.

$\diamond$

References

[Bei78] A. A. Beilinson, Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra, *Functional Analysis and Its Applications* **12** (1978), 214–216.

[Orl92] D. O. Orlov, Projective bundles, monoidal transformations, and derived categories of coherent sheaves, *Izv. Ross. Akad. Nauk Ser. Mat.* **56** (1992), no. 4, 852–862 (English translation: *Russian Acad. Sci. Izv. Math.* **41** (1993), no. 1, 133–141).