Fano 2-513-105-a. $\mathscr{Z}(\mathbb{P}^2 \times \mathbb{P}^4, \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(1))$.

Invariants. $h^0(-K) = 105, \quad (-K)^4 = 513, \quad h^{1,1} = 2, \quad h^{2,2} = 3, \quad -\chi(T) = -18.$

Description. Let h, H denote the hyperplane classes of $\mathbb{P}^2, \mathbb{P}^4$ restricted to X, respectively, and let π_i be the restriction to X of the projection forgetting the i-th factor of the displayed product.

(1) $\pi_1 \colon X \to \mathbb{P}^4$ is the blow-up of a line $L \subset \mathbb{P}^4$ (the Fano fourfold 1-625-126-a). It is an elementary extremal divisorial contraction.
(2) $\pi_2 \colon X \to \mathbb{P}^2$ is the $\mathbb{P}^2$-bundle $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)) \to \mathbb{P}^2$. It is an elementary extremal contraction of fiber type.

- $\mathrm{Nef}(X) = \langle h, H \rangle$ is simplicial, and the two projections give its two extremal contractions.
- X is toric, rational, and unique up to isomorphism.

Semiorthogonal decompositions. Orlov's blow-up and projective bundle formulas [2] give

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(L), \mathrm{D}^{\mathrm{b}}(L), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^4) \rangle, \qquad \mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2) \otimes \mathcal{O}(H), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2) \otimes \mathcal{O}(2H) \rangle,$$

where the components are understood with their standard fully faithful embeddings.

Explanation. Write $\mathbb{P}^2 = \mathbb{P}(V_3)$ and $\mathbb{P}^4 = \mathbb{P}(V_5)$, parametrizing lines. Since $H^0(\mathbb{P}^2, \mathcal{Q}_{\mathbb{P}^2}) = V_3$, the section corresponds to a general surjection $V_5 \to V_3$. Its kernel defines the line $L \subset \mathbb{P}^4$.

Description of π_1. For the ambient projection $p_1 \colon \mathbb{P}^2 \times \mathbb{P}^4 \to \mathbb{P}^4$, the relative pushforward is

$$p_{1*}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(1)) = V_3 \otimes \mathcal{O}_{\mathbb{P}^4}(1), \qquad R^i p_{1*}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(1)) = 0 \quad (i > 0).$$

The induced section is given by three independent linear forms, whose common zero scheme is the smooth line L, of codimension three. Thus [1, Lemma 3.1] applies and gives $X \simeq \mathrm{Bl}_L \mathbb{P}^4$, with exceptional divisor $E \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Adjunction and the canonical bundle formula for the blow-up give, respectively,

$$-K_X = 2h + 3H = 5H - 2E, \qquad E = H - h.$$

Description of π_2. For the ambient projection $p_2 \colon \mathbb{P}^2 \times \mathbb{P}^4 \to \mathbb{P}^2$,

$$p_{2*}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(1)) = \mathcal{Q}_{\mathbb{P}^2} \otimes V_5^\vee, \qquad R^i p_{2*}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}^4}(1)) = 0 \quad (i > 0).$$

The induced section defines the composition

$$\varphi \colon V_5 \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow V_3 \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{Q}_{\mathbb{P}^2}.$$

Both arrows are surjective, hence φ is surjective and, scheme-theoretically, $X = \mathbb{P}_{\mathbb{P}^2}(\ker \varphi)$. Choosing a splitting of the surjection $V_5 \to V_3$ and using the tautological sequence, we obtain directly

$$\ker \varphi \simeq \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1).$$

This proves the asserted projective bundle description.

Since X is the blow-up of $\mathbb{P}^4$ along a smooth connected center, $\rho(X) = 2$. The independent nef classes H and h vanish on curves contracted by π_1 and π_2, respectively, so they generate the two boundary rays of $\mathrm{Nef}(X)$. The projective bundle description makes X toric and rational, while the action on surjections $V_5 \to V_3$ is transitive, proving uniqueness up to isomorphism. ◇

References

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[2] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.