Fano 2-513-105-b. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2,5),\ \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee\big)$.

Invariants. $h^0(-K) = 105$, $(-K)^4 = 513$, $h^{1,1} = 2$, $h^{2,1} = 0$, $h^{3,1} = 0$, $h^{2,2} = 3$, $-\chi(T) = -18$.

Description.

(1) $\pi_1 \colon X \to \mathrm{Gr}(2,5)$ is an extremal small contraction onto a Schubert variety of degree 3, the cone over $\mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}^5$: it contracts a plane with normal bundle $\mathcal{O}(-1)^{\oplus 2}$ to the vertex, and is an isomorphism elsewhere.

(2) $\pi_2 \colon X \to \mathbb{P}^2$ is an extremal contraction of fiber type, the $\mathbb{P}^2$-bundle $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2})$.

- h, L denote the pullbacks of the hyperplane class of $\mathbb{P}^2$ and of the Plücker class.
- $\mathrm{Nef}(X) = \langle h, L\rangle$ is simplicial.
- The flip of π_1 replaces the plane by a line with normal bundle $\mathcal{O}(-1)^{\oplus 3}$.
- X is toric, and $\mathrm{Aut}(X)$ has dimension 18.
- X is rational.

Semiorthogonal decompositions.

- From π_2 and [Orl]: $\mathrm{D^b}(X) = \langle \pi_2^* \mathrm{D^b}(\mathbb{P}^2), \pi_2^* \mathrm{D^b}(\mathbb{P}^2)(L), \pi_2^* \mathrm{D^b}(\mathbb{P}^2)(2L)\rangle$, a full exceptional collection of length 9.

Explanation.

Write $\mathbb{P}^2 = \mathbb{P}(V_3)$ and $\mathrm{Gr}(2,5) = \mathrm{Gr}(2,V_5)$. As in [BFMT, Proposition 3.3], the section of $\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee$ is a general surjection $\varphi \colon V_5 \to V_3$, with kernel a plane K, and vanishes at (x,U) if and only if $\varphi(U) \subset x$, that is $U \subset U_3(x) := \varphi^{-1}(x)$, a 3-space containing K. Thus

$$X = \{(x,U) :\ U \subset U_3(x)\}.$$

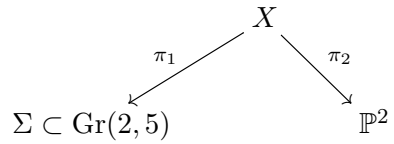
Description of π_1. The fiber of π_1 over U is $\{x : \varphi(U) \subset x\}$. It is empty if $\varphi(U)$ has dimension 2, the point $\varphi(U)$ if $U \cap K$ is a line, and all of $\mathbb{P}^2$ if $U = K$. Since $\varphi|_U$ has constant rank one on $\{U : U \cap K \text{ is a line}\}$, the map $U \mapsto (\varphi(U), U)$ is a morphism there, inverse to π_1. So π_1 is birational onto the Schubert variety $\Sigma = \{U : U \cap K \neq 0\}$, of class σ_2 and degree 3, the cone with vertex $[K]$ over $\mathbb{P}(K) \times \mathbb{P}(V_5/K) \simeq \mathbb{P}^1 \times \mathbb{P}^2$; it contracts exactly the plane $\Pi = \mathbb{P}^2 \times \{K\}$, and is an isomorphism elsewhere. A line $\ell \subset \Pi$ has $h \cdot \ell = 1$ and $L \cdot \ell = 0$, and the nef class L vanishes exactly on the curves of Π; so π_1 is the contraction of an extremal ray, which is K_X-negative since X is Fano. The normal bundle of Π is $\mathcal{H}om(K, U_3(x)/K) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$, and [Kaw] gives the flip. Explicitly, the other small resolution of Σ is $X^+ = \{(p,U) :\ p \subset U \cap K\}$, a $\mathbb{P}^3$-bundle over $\mathbb{P}(K) \simeq \mathbb{P}^1$, in which the fiber over $[K]$ is the line $\{(p,K)\}$.

Description of π_2. The fiber of π_2 over x is $\mathrm{Gr}(2, U_3(x)) \simeq \mathbb{P}^2$, on which L is the hyperplane class. As $U_3(x)$ is an extension of $\mathcal{O}(-1)$ by $K \otimes \mathcal{O}$, which splits, $X = \mathrm{Gr}(2, K \otimes \mathcal{O} \oplus \mathcal{O}(-1)) \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2})$, and Π is the section given by $U = K$. A line f in a fiber has $L \cdot f = 1$ and $h \cdot f = 0$, and the nef class h vanishes exactly on the curves of the fibers.

Cones. The classes h, L are nef, and ℓ, f form the dual basis:

	ℓ	f
h	1	0
L	0	1

Hence $\mathrm{Nef}(X) = \langle h, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle \ell, f\rangle$ are dual simplicial cones, and π_1, π_2 are the two extremal contractions.



Automorphisms and comparison. Being a projectivized split bundle over $\mathbb{P}^2$, X is toric and rational, with $h^{2,2} = 3$. Every automorphism preserves both contractions, so $\mathrm{Aut}(X)$ is the stabilizer of φ in $(\mathrm{GL}(V_3) \times \mathrm{GL}(V_5))/\mathbb{C}^*$; as the surjections form an open orbit, it has dimension $9 + 25 - 1 - 15 = 18$. The blow-up of $\mathbb{P}^4$ along a line (2-513-105-a) has the same invariants and the same base $\mathbb{P}^2$, but its other contraction is divisorial, while π_1 is small; so the two are not isomorphic.

◇

References

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