Fano 2-544-111-a. $\mathscr{Z}\big(\mathrm{Fl}(1,2,5),\ \bigwedge^2 \mathcal{Q}^\vee \otimes (\mathcal{U}_2/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee\big)$, where $\mathcal{Q} = V_5/\mathcal{U}_2$ is the rank-three quotient bundle.

Invariants. $h^0(-K) = 111, \quad (-K)^4 = 544, \quad h^{1,1} = 2, \quad h^{2,2} = 2, \quad -\chi(T) = -20.$

Description. Regard $\mathrm{Fl}(1,2,5)$ as the incidence subvariety of $\mathbb{P}(V_5) \times \mathrm{Gr}(2,5)$, and let π_i forget the i-th factor. Let $v \in V_5$ be a nonzero vector and put $p = [v] \in \mathbb{P}^4$. Write $H = c_1(\mathcal{U}_1^\vee|_X)$ and $L = c_1(\det \mathcal{U}_2^\vee|_X)$.

(1) $\pi_1 : X \to \mathbb{P}^3 \subset \mathrm{Gr}(2,5)$ is the K-negative extremal $\mathbb{P}^1$-bundle $X \simeq \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(-1))$.
(2) $\pi_2 : X \simeq \mathrm{Bl}_p \mathbb{P}^4 \to \mathbb{P}^4$ is the K-negative extremal divisorial contraction onto the Fano fourfold 1-625-126-a.

- $-K_X = 2H + 3L$, and $\mathrm{Nef}(X) = \langle H, L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions. By Orlov's blow-up formula [2],

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(\mathrm{pt}), \mathrm{D}^{\mathrm{b}}(\mathrm{pt}), \mathrm{D}^{\mathrm{b}}(\mathrm{pt}), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^4)\rangle,$$

with the standard fully faithful embeddings and twists. In particular, $\mathrm{D}^{\mathrm{b}}(X)$ admits a full exceptional collection of length eight.

Explanation. Since $\det \mathcal{Q} = \det \mathcal{U}_2^\vee$, the defining bundle is

$$\bigwedge^2 \mathcal{Q}^\vee \otimes (\mathcal{U}_2/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee \simeq \mathcal{Q} \otimes \det \mathcal{Q}^\vee \otimes \det \mathcal{U}_2^\vee \simeq \mathcal{Q}.$$

For the projection $q : \mathrm{Fl}(1,2,5) \to \mathrm{Gr}(2,5)$, $q_*\mathcal{Q} = \mathcal{Q}$ and $R^i q_* \mathcal{Q} = 0$ for $i > 0$. The section is therefore determined by $v \in V_5$ and

$$X = \{(U_1, U_2) : U_1 \subset U_2,\ v \in U_2\}.$$

Description of π_1. The zero locus of the induced section of $\mathcal{Q}$ on $\mathrm{Gr}(2,5)$ is $Z = \{U_2 : v \in U_2\} \simeq \mathbb{P}(V_5/\langle v\rangle) \simeq \mathbb{P}^3$. On Z we have

$$0 \longrightarrow \mathcal{O}_Z \xrightarrow{v} \mathcal{U}_2|_Z \longrightarrow \mathcal{O}_Z(-1) \longrightarrow 0.$$

Since $H^1(\mathbb{P}^3, \mathcal{O}(1)) = 0$, this sequence splits. Thus the restriction of the flag projection is

$$X = \mathbb{P}_Z(\mathcal{U}_2|_Z) \simeq \mathbb{P}_{\mathbb{P}^3}(\mathcal{O} \oplus \mathcal{O}(-1)),$$

and L is the pullback of the hyperplane class of Z.

Description of π_2. Over $\mathbb{P}(V_5)$ the flag variety is $\mathbb{P}_{\mathbb{P}^4}(V_5/\mathcal{U}_1)$, and the defining section is induced by the composite

$$\mathcal{O}_{\mathbb{P}^4} \xrightarrow{v} V_5/\mathcal{U}_1 \longrightarrow V_5/\mathcal{U}_2.$$

The first arrow is a section of the rank-four bundle $V_5/\mathcal{U}_1 \simeq T_{\mathbb{P}^4}(-1)$, with a single regular zero at p. The blow-up form of the Cayley trick [1] identifies its incidence zero locus with $\mathrm{Bl}_p \mathbb{P}^4$. The exceptional fibre is $\mathbb{P}(V_5/\langle v\rangle) \simeq \mathbb{P}^3$. On X the vector v induces a section of $\mathcal{U}_2/\mathcal{U}_1$, vanishing precisely along E; hence

$$E = c_1(\mathcal{U}_2/\mathcal{U}_1) = H - L, \qquad -K_X = 5H - 3E = 2H + 3L.$$

The two classes H and L are nef pullbacks from the targets. They vanish, respectively, on the lines in E and on the fibres of π_1, and therefore generate the nef cone since $\rho(X) = 2$. Both contractions are K-negative by the formula for $-K_X$. Rationality follows from $X \simeq \mathrm{Bl}_p \mathbb{P}^4$. ◇

References

[1] M. Bernardara, E. Fatighenti, L. Manivel and F. Tanturri, *Fano fourfolds of K3 type*, Ann. Inst. Fourier (2026), doi:10.5802/aif.3761.

[2] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), no. 1, 133–141.