Fano 2-70-21-c. $\mathscr{Z}(\mathbb{P}^1 \times \mathrm{Gr}(2,6), \mathcal{O}(0,1)^{\oplus 4} \oplus \mathcal{O}(1,1))$.

Invariants. $h^0(-K) = 21, \quad (-K)^4 = 70, \quad h^{1,1} = 2, \quad h^{3,1} = 1, \quad h^{2,2} = 28, -\chi(T) = 27.$

Description. **This is K3-25 from** [1]**.** Let $Y = \mathrm{Gr}(2,6) \cap H_1 \cap H_2 \cap H_3 \cap H_4$ be the smooth Mukai fourfold of genus 8 (Fano 1-224-51-a), and let $S = Y \cap H_5 \cap H_6$ be a smooth K3 surface of genus 8. The factors are ordered as displayed, and π_i omits factor i.

(1) $\pi_1 : X \to Y$ is an elementary K-negative divisorial contraction, $X \simeq \mathrm{Bl}_S Y$.
(2) $\pi_2 : X \to \mathbb{P}^1$ is an elementary K-negative contraction of fiber type, a flat pencil of hyperplane sections of Y, with general fiber a prime Fano threefold of genus 8 and degree 14. For general X, exactly 48 fibers have one ordinary double point, and all remaining fibers are smooth.

- $\mathrm{Nef}(X) = \langle H, L \rangle$ is simplicial, where $H = \pi_1^* \mathcal{O}_Y(1)$ and $L = \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$.
- X is rational.

Semiorthogonal decompositions. By Orlov's codimension-two blow-up formula [4, Theorem 4.3],

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S), \mathrm{D}^{\mathrm{b}}(Y) \rangle,$$

where the components are embedded by the usual fully faithful functors. For a general codimension-four linear section Y of $\mathrm{Gr}(2,6)$, [3, Corollary 10.3] gives a full exceptional collection of length 12 on Y. Thus $\mathrm{D}^{\mathrm{b}}(X)$ admits a semiorthogonal decomposition consisting of $\mathrm{D}^{\mathrm{b}}(S)$ and 12 exceptional objects.

Explanation.

Put $G = \mathrm{Gr}(2,6)$, and denote by $\mathcal{U}$ its rank-two tautological subbundle, with $0 \to \mathcal{U} \to \mathcal{O}_G^{\oplus 6} \to \mathcal{Q} \to 0$. Write $\mathcal{O}_G(1) = \det \mathcal{U}^\vee$ and let H also denote its restriction to Y and its pullback to X. The four independent sections of $\mathcal{O}(0,1)$ cut out a smooth codimension-four linear section Y; Lefschetz and adjunction give $\mathrm{Pic}(Y) = \mathbb{Z}H$, $-K_Y = 2H$ and $H^4 = 14$. The fifth equation is a general section of $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Y(1)$.

Description of π_1**.** Let $p : \mathbb{P}^1 \times Y \to Y$. Since $\mathcal{Q}_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(1)$, the last equation is a section of $\mathcal{Q}_{\mathbb{P}^1} \boxtimes \mathcal{O}_Y(1)$. The projective bundle formula gives

$$p_*(\mathcal{Q}_{\mathbb{P}^1} \boxtimes \mathcal{O}_Y(1)) \simeq \mathcal{O}_Y(1)^{\oplus 2}, \qquad R^j p_*(\mathcal{Q}_{\mathbb{P}^1} \boxtimes \mathcal{O}_Y(1)) = 0 \quad (j > 0).$$

Thus the section corresponds to two general hyperplane sections $s_0, s_1 \in H^0(Y, \mathcal{O}_Y(1))$. Since $\mathcal{O}_Y(1)$ is globally generated, the codimension-two Cayley trick [1, Lemma 3.1] applies and yields

$$X \simeq \mathrm{Bl}_S(Y), \qquad S = \mathscr{Z}(Y, \mathcal{O}_Y(1)^{\oplus 2}) = Y \cap H_5 \cap H_6.$$

Scheme-theoretically, the center is

$$S = D_0\left(\mathcal{O}_Y(-1) \xrightarrow{(s_0, s_1)} \mathcal{O}_Y^{\oplus 2}\right), \qquad [S] = c_2(\mathcal{O}_Y(1)^{\oplus 2}) = H^2.$$

By Bertini, S is smooth of codimension two. Its normal bundle and the exceptional divisor are

$$N_{S/Y} \simeq \mathcal{O}_S(1)^{\oplus 2}, \qquad E = \mathbb{P}_S(N_{S/Y}) \simeq S \times \mathbb{P}^1.$$

Since $S = \mathrm{Gr}(2,6) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5 \cap H_6$, adjunction and Lefschetz give $K_S = 0$ and $H^1(S, \mathcal{O}_S) = 0$. Finally, $H_S^2 = 14$, so S is a polarized K3 surface of genus 8.

Description of π_2. Let $q : \mathbb{P}^1 \times Y \to \mathbb{P}^1$. Since $\mathcal{O}_Y(1) = \mathcal{O}_Y(K_Y + 3H)$ and $3H$ is ample, Kodaira vanishing and the projection formula give

$$q_*(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_Y(1)) \simeq \mathcal{O}_{\mathbb{P}^1}(1) \otimes H^0(Y, \mathcal{O}_Y(1)), \qquad R^j q_* = 0 \quad (j > 0).$$

The actual map corresponding to the section is the composite

$$\mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{(s_0,s_1)} H^0(Y, \mathcal{O}_Y(1)) \otimes \mathcal{O}_{\mathbb{P}^1 \times Y} \xrightarrow{\text{ev}} \mathcal{O}_Y(1),$$

where the first arrow sends $[u : v]$ to $us_0 + vs_1$. Consequently the scheme-theoretic fiber over $[u : v]$ is

$$X_{[u:v]} = Y \cap V(us_0 + vs_1) = G \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap V(us_0 + vs_1).$$

The linear independence of s_0, s_1 ensures that every member is an effective Cartier divisor in Y. Its class H is primitive in $\mathrm{Pic}(Y)$, so every member is integral. In particular, the relative Cartier divisor is flat over $\mathbb{P}^1$ and its fibers have pure dimension 3. Bertini shows that the general one W is smooth, while adjunction yields $-K_W = H|_W$ and $H_W^3 = 14$: it is a prime Fano threefold of genus 8.

The number of singular fibers can be obtained from Euler characteristics, without using the first-jet bundle. A general pencil of hyperplane sections of the smooth fourfold Y is a Lefschetz pencil: its singular members each have one ordinary double point, and their critical values are distinct [5, Exp. XVII]. Let N be their number and W a smooth member. Since $X = \mathrm{Bl}_S Y$ and S is a K3 surface,

$$e(X) = e(Y) + e(S) = e(Y) + 24.$$

On the other hand, a nodal threefold has Euler characteristic one larger than that of its smoothing, so the fibration π_2 gives

$$e(X) = 2e(W) + N.$$

It remains to calculate $e(Y)$ and $e(W)$. For this we use the natural Chern classes of the tautological bundle on the Grassmannian. Set $\eta = c_2(\mathcal{U}^\vee)$, and use the same notation for its restriction to Y. Recall that $H = c_1(\mathcal{U}^\vee)$ is the Plücker class. The intersection products required below are

$$\int_Y H^4 = 14, \qquad \int_Y H^2\eta = 5, \qquad \int_Y \eta^2 = 2.$$

Indeed, Y is cut out by four Plücker hyperplanes, so its class in G is H^4. These numbers are therefore ordinary intersection products on $\mathrm{Gr}(2, 6)$, obtained by Schubert calculus.

To compute the Chern classes, we may temporarily write x, y for the formal Chern roots of $\mathcal{U}^\vee$; thus $H = x + y$ and $\eta = xy$. Since $T_G = \mathcal{U}^\vee \otimes \mathcal{Q}$, and Y is the intersection of four hyperplanes, the normal sequence gives

$$c_t(T_Y) = \frac{(1 + xt)^6(1 + yt)^6}{(1 - (x - y)^2t^2)(1 + Ht)^4}.$$

Expanding to degree four, we find

$$\begin{aligned} c_1(Y) &= 2H, & c_2(Y) &= 2H^2 + 2\eta, \\ c_3(Y) &= 2H^3 - 2H\eta, & c_4(Y) &= 2H^4 - 6H^2\eta + 7\eta^2, \end{aligned}$$

and consequently

$$e(Y) = \int_Y c_4(Y) = 2 \cdot 14 - 6 \cdot 5 + 7 \cdot 2 = 12.$$

For a smooth hyperplane section $W \subset Y$, the normal sequence gives

$$c_3(T_W) = (c_3(Y) - Hc_2(Y) + H^2c_1(Y) - H^3)|_W = (H^3 - 4H\eta)|_W.$$

It follows that

$$e(W) = \int_Y H(H^3 - 4H\eta) = 14 - 4 \cdot 5 = -6,$$

and

$$N = e(Y) + 24 - 2e(W) = 12 + 24 + 12 = 48.$$

Thus the discriminant is a reduced subscheme of $\mathbb{P}^1$ of degree 48, and every singular fiber has exactly one ordinary double point.

The nef and Mori cones. The blow-up description gives $\rho(X) = 2$. The pullbacks $H = \pi_1^*\mathcal{O}_Y(1)$ and $L = \pi_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ are nef. If f is a line in an exceptional fiber of π_1, and C is a curve in a fiber of π_2, their intersections are

	f	C
H	0	$H \cdot C > 0$
L	1	0

Thus H and L lie on distinct boundary rays of the nef cone; since $\rho(X) = 2$, they generate the whole cone:

$$\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}H + \mathbb{R}_{\geq 0}L.$$

By duality, the Mori cone is

$$\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}[f] + \mathbb{R}_{\geq 0}[C].$$

Adjunction gives $-K_X = H + L$, so $K_X \cdot f = -1$ and $K_X \cdot C = -H \cdot C < 0$. Both rays are therefore K-negative and extremal, and π_1 and π_2 are the corresponding elementary Mori contractions. The exceptional divisor has class $E = H - L$.

The Mukai fourfold Y is rational by [2, Theorem 3.3], so its blow-up X is rational. This argument uses the rationality of the *fourfold* Y, not that of its generally irrational prime Fano threefold hyperplane sections.

◇

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