Fano 2-80-23-c. $X = \mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Gr}(3,6), \bigwedge^2 \mathcal{U}^\vee \oplus \mathcal{O}(0,1)^{\oplus 2} \oplus \mathcal{O}(1,1)\big)$.

Invariants. $h^0(-K_X) = 23$, $(-K_X)^4 = 80$, $h^{1,1} = 2$, $h^{2,1} = 0$, $h^{3,1} = 1$, $h^{2,2} = 24$, and $-\chi(T_X) = 23$.

Description. **This is the family K3–26 in** [1, example K3–26]. The ambient space is $\mathbb{P}^1 \times \mathrm{Gr}(3, V)$, and π_i always omits factor i.

(1) $\pi_1 : X \to Y$ is an elementary K-negative divisorial contraction, $X \simeq \mathrm{Bl}_S Y$, where $Y \subset \mathrm{Gr}(3,6)$ is the index-two del Pezzo fourfold X_{16}, and S a polarized K3 surface of genus 9 and degree 16.

(2) $\pi_2 : X \to \mathbb{P}^1$ is an elementary Mori fiber space; its general fiber is a smooth prime Fano threefold W_{16} of index one, degree 16 and genus 9.

- Write $L = \pi_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $H = \pi_1^*\mathcal{O}_Y(1)$, using the same letter for the pullback.
- $\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}H + \mathbb{R}_{\geq 0}L$ is simplicial; $-K_X = H + L$, $E = H - L$.
- Rationality is not established by the descriptions above (and is recorded as unknown in [1]).
- The family admits a multiplicative Chow–Künneth decomposition [2, §4.11].

Semiorthogonal decompositions. Orlov's blow-up formula [4] gives $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(S),\ \pi_1^*\,\mathrm{D}^{\mathrm{b}}(Y)\rangle$, where the first component is embedded by the usual exceptional-divisor functor. The full exceptional collection of length 8 on Y therefore yields $\mathrm{D}^{\mathrm{b}}(S)$ together with 8 exceptional objects [1, 2].

Explanation.

Proof. Let V be six-dimensional, endowed with a general $\omega \in \bigwedge^2 V^\vee$ of maximal rank. On $\mathrm{Gr}(3,V)$, ω defines a section of $\bigwedge^2 \mathcal{U}^\vee$, whose scheme-theoretic zero locus is the smooth 6-fold $G = \mathrm{LG}(3,V)$, of expected codimension 3. The remaining two Plücker sections cut out Y, smooth of dimension 4 by Bertini. The Plücker polarization H generates $\mathrm{Pic}(Y) \simeq \mathbb{Z}$ by Lefschetz; $K_G = -4H$ and $K_Y = -2H$ by adjunction. The classical Plücker degree of G is 16.

The last summand $\mathcal{O}(1,1)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes H$ on $\mathbb{P}^1 \times Y$. For homogeneous coordinates $[t_0 : t_1]$ on $\mathbb{P}^1$, its section is uniquely a pencil

$$\widetilde{s} = t_0 s_0 + t_1 s_1, \qquad s_0, s_1 \in H^0(Y, H), \quad X = \{(t,y) : \widetilde{s}(t,y) = 0\}. \tag{1}$$

The Künneth formula identifies global sections with pairs, and the restriction map from Plücker linear forms on $\mathrm{Gr}(3,V)$ to $H^0(Y,H)$ is surjective, since Y is a linear section. Thus s_0, s_1 are an arbitrary general pair on Y; no extra condition is imposed by the original homogeneous model. The section of $\mathcal{O}(1,1)$ is consequently regular and X is smooth. By adjunction,

$$-K_X = ((2L + 2H) - (L + H))|_X = L + H. \tag{2}$$

Description of π_1. Write $p : \mathbb{P}^1 \times Y \to Y$. The relative projective-bundle formula, with all higher direct images vanishing, gives

$$p_*(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes H) \simeq H^{\oplus 2}, \qquad R^i p_*(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes H) = 0 \quad (i > 0). \tag{3}$$

Under the evaluation $H^0(\mathbb{P}^1, \mathcal{O}(1)) \otimes \mathcal{O}_Y \to p_*\mathcal{O}(1,0)$, the pushed-forward section of (1) is precisely

$$\sigma : \mathcal{O}_Y \xrightarrow{(s_0, s_1)} H^{\oplus 2}. \tag{4}$$

Its scheme-theoretic rank-zero locus is $S = D_0(\sigma) = Z_Y(s_0, s_1)$, with ideal $\mathcal{I}_{S/Y} = (s_0, s_1)$ locally after trivializing H. It has the expected and actual codimension 2. Since the induced pair is general in $H^0(Y,H)^{\oplus 2}$, Bertini implies that S is smooth and the section is regular. The geometric Chern-class, normal-bundle and adjunction identities are

$$\begin{aligned} &[S] = c_2(H^{\oplus 2}) = h^2 \in A^2(Y), \quad N_{S/Y} \simeq H|_S^{\oplus 2}, \quad c(N_{S/Y}) = (1 + h|_S)^2, \\ &K_S = (K_Y + 2H)|_S = 0. \end{aligned} \tag{5}$$

where $h = c_1(H)$. By Lefschetz $H^1(S, \mathcal{O}_S) = 0$; hence S is a K3 surface. Moreover $h|_S^2 = 16$, so the primitive polarization has genus 9. Its embedding is the codimension-four linear section $G \cap H_1 \cap H_2 \cap H_3 \cap H_4$ in the Plücker projective space. The rank-two Cayley trick (see [1, §3.1]) identifies X *scheme-theoretically*, with $\mathrm{Bl}_S Y$. Thus π_1 is a blow-up and its exceptional divisor is $E = \mathbb{P}_S(N_{S/Y}) \simeq S \times \mathbb{P}^1$. The divisor relation $L = H - E$ follows from the strict transforms of members of the pencil; the canonical formula $-K_X = 2H - E = H + L$ agrees with (2). Since $\rho(X) = 2$ and $\rho(Y) = 1$, this is elementary; its fibers over S are curves with $(-K_X)\cdot e = 1$, hence it is a K-negative divisorial extremal contraction.

Description of π_2. Let $q : \mathbb{P}^1 \times Y \to \mathbb{P}^1$. By Kodaira vanishing ($H = K_Y + 3H$), cohomology and base change give

$$q_*(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes H) \simeq \mathcal{O}_{\mathbb{P}^1}(1) \otimes H^0(Y, H), \qquad R^i q_*(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes H) = 0 \quad (i > 0). \tag{6}$$

The induced map is the section

$$\mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \otimes H^0(Y, H), \qquad 1 \longmapsto t_0 s_0 + t_1 s_1.$$

Consequently the scheme-theoretic fiber over $t = [t_0 : t_1]$ is $Y \cap Z(t_0 s_0 + t_1 s_1)$, a hyperplane section of Y in its Plücker embedding. For general t it is smooth by Bertini. Since $K_Y = -2H$, adjunction yields $-K_{X_t} = H|_{X_t}$; its degree is $H_Y^4 = 16$. Thus X_t is a prime Fano threefold W_{16} of genus 9. No fiber can split into two effective divisors, since $\mathrm{Pic}(Y) = \mathbb{Z}H$ and H is primitive; in particular π_2 has connected fibers. The fibers are of dimension three and $-K_X|_{X_t} = H|_{X_t}$ is ample. As $\rho(X/\mathbb{P}^1) = 1$, π_2 is an elementary K-negative Mori fiber space.

For clarity about singular fibers, the incidence scheme of critical points is defined on $\mathbb{P}^1 \times Y$ by the *relative first jet*

$$j^1_{/\mathbb{P}^1}(\widetilde{s}) : \mathcal{O}_{\mathbb{P}^1 \times Y} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes J^1_Y(H), \tag{7}$$

where $J^1_Y(H)$ has rank 5 and fits into

$$0 \longrightarrow \Omega^1_Y \otimes H \longrightarrow J^1_Y(H) \longrightarrow H \longrightarrow 0.$$

Its zero locus imposes *both* $\widetilde{s} = 0$ and $d_Y \widetilde{s} = 0$; its image in $\mathbb{P}^1$ is the singular-fiber discriminant, while the zero locus itself is the critical locus in X. Because H is very ample, its global sections generate first jets, hence the induced linear system generates the bundle in (7). For a general pencil the jet section is transverse, so the critical scheme is finite and reduced (possibly empty); its expected codimension is 5, and its class is

$$[\mathrm{Crit}(\pi_2)] = c_5(\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes J^1_Y(H)) = L\, c_4(J^1_Y(H)), \qquad c(J^1_Y(H)) = c(\Omega^1_Y \otimes H)(1 + h), \tag{8}$$

see for example [3, Theorem 5.2]. The hyperplane pencil is general: there is no additional speciality coming from the original bundle presentation.

Nef cone. Finally, both H and L are nef as pullbacks of ample classes, and they are independent. For an exceptional line $e \subset E$, one has $H \cdot e = 0$, $L \cdot e = 1$; for a curve f in a fiber of π_2, one has $L \cdot f = 0$, $H \cdot f > 0$. These two curve classes support distinct boundary rays of the two-dimensional nef cone. Since $\rho(X) = 2$, they exhaust its boundary: $\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}H + \mathbb{R}_{\geq 0}L$, simplicial. The identification with K3–26, including the blow-up and Fano-threefold fibration, is established in [1, example K3–26]; the Chow-theoretic consequence is [2, §4.11].

◇

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