Fano 2-90-25-d. $\mathscr{Z}(\mathbb{P}^1 \times \mathrm{Gr}(2,7), \mathcal{O}(0,1) \oplus \mathcal{O}(1,1) \oplus (\mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{Q}^\vee(1)))$, where $\mathcal{Q}$ is the rank-five tautological quotient bundle on $\mathrm{Gr}(2,7)$ and $\mathcal{O}_{\mathrm{Gr}(2,7)}(1) = \det \mathcal{U}^\vee$.

Invariants. $h^0(-K) = 25$, $(-K)^4 = 90$, $h^{1,1} = 2$, $h^{2,1} = 0$, $h^{3,1} = 1$, $h^{2,2} = 22$, $-\chi(T) = 21$.

Description. **This is the family K3–28 in** [1, example K3–26]. Let π_i denotes the projection omitting the ith factor of the ambient space $\mathbb{P}^1 \times \mathrm{Gr}(2,7)$.

(1) $\pi_1 : X \to W$ is an elementary K-negative divisorial contraction, $X \simeq \mathrm{Bl}_S W$, the blow-up of $W \subset \mathrm{Gr}(2,7)$, the Mukai fourfold of genus 10, along S, a K3 surface of genus 10 and degree 18.

(2) $\pi_2 : X \to \mathbb{P}^1$ is an elementary Mori fibre space; its general fibre is a smooth prime Fano threefold of genus 10 and degree 18. For a general X its discriminant has degree 30: there are 30 singular fibres, each with exactly one ordinary double point.

- If H is the pullback of $\mathcal{O}_W(1)$ and $A = \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)$, then $\mathrm{Nef}(X) = \mathbb{R}_{\geq 0} H + \mathbb{R}_{\geq 0} A$ is simplicial.
- X is rational [2, Theorem 3.3].

Semiorthogonal decompositions.

- The divisorial description gives the Orlov component $D^{\mathrm{b}}(S)$ and six exceptional objects; the fibration description gives four exceptional objects and an unidentified residual category, as recorded in [1].

Explanation.

Proof. Let $V = \mathbb{C}^7$. The tautological sequence on $\mathrm{Gr}(2, V)$ is

$$0 \longrightarrow \mathcal{U} \longrightarrow V \otimes \mathcal{O} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

A section of $\mathcal{Q}^\vee(1)$ is induced by a three-form $\varphi \in \bigwedge^3 V^\vee$: on a two-plane $U \subset V$ it is $\varphi(u_1, u_2, -) \in (V/U)^\vee \otimes \det(U)^\vee$. For a general stable three-form, its zero locus is the smooth five-dimensional adjoint G_2-variety Σ, in its Plücker embedding of degree 18 and index 3; see [1, 3]. The remaining two equations give a hyperplane $W = \Sigma \cap H_0$ and a general pencil of hyperplanes on W. Indeed, Σ is linearly normal in its $\mathbb{P}^{13}$-span, and the restriction $H^0(\Sigma, \mathcal{O}_\Sigma(1)) \to H^0(W, \mathcal{O}_W(1))$ is surjective since $H^1(\Sigma, \mathcal{O}_\Sigma) = 0$. Thus the pencil induced from the Grassmannian is general on W. In particular $K_W = -2H_W$, and $S = \Sigma \cap H_0 \cap H_1 \cap H_2$ is a smooth K3 surface with $\mathcal{O}_S(1)^2 = 18$.

Description of π_1. Let $s_1, s_2 \in H^0(W, \mathcal{O}_W(1))$ be the two independent sections of the pencil. Along $\mathbb{P}^1 \times W$ the remaining equation is

$$\mathcal{O}_{\mathbb{P}^1 \times W}(-1, 0) \xrightarrow{a s_1 + b s_2} \mathcal{O}_{\mathbb{P}^1 \times W}(0, 1),$$

where $[a : b]$ are homogeneous coordinates on $\mathbb{P}^1$. Equivalently, the induced map on W is

$$\psi : \mathcal{O}_W(-1) \xrightarrow{(s_1, s_2)} \mathcal{O}_W^{\oplus 2}, \quad S = D_0(\psi), \quad [S] = c_2(\mathcal{O}_W(1)^{\oplus 2}) = H_W^2,$$

with $N_{S/W} = \mathcal{O}_S(1)^{\oplus 2}$ and expected and actual codimension two, since the two hyperplanes are general. The projective formula gives

$$(\mathrm{pr}_W)_* \mathcal{O}_{\mathbb{P}^1 \times W}(1, 1) = \mathcal{O}_W(1)^{\oplus 2}, \qquad R^i (\mathrm{pr}_W)_* \mathcal{O}(1, 1) = 0 \quad (i > 0).$$

Applying the codimension-two Cayley trick [1, Section 3] to this regular pair yields $X \simeq \mathrm{Bl}_S W$. The exceptional divisor is $E = H - A$, and adjunction gives $-K_X = 2H - E = H + A$. Since $\rho(W) = 1$ by the Lefschetz theorem and S is connected, $\rho(X) = 2$; π_1 is an elementary extremal divisorial contraction.

Description of π_2. The hyperplane section W is embedded by the complete linear system $|H_W|$, and s_1, s_2 span a general pencil in this system. Set $p = \pi_2 : X \to \mathbb{P}^1$; then

$$F_{[a:b]} = \mathscr{Z}_W(as_1 + bs_2) = \Sigma \cap H_0 \cap H_{[a:b]}.$$

The relative equation is the evaluation of the canonical composite

$$\mathrm{pr}_{\mathbb{P}^1}^*\,\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow H^0(W, \mathcal{O}_W(1)) \otimes \mathcal{O}_{\mathbb{P}^1 \times W} \xrightarrow{\mathrm{ev}} \mathrm{pr}_W^*\,\mathcal{O}_W(1).$$

Here $(\mathrm{pr}_{\mathbb{P}^1})_*\,\mathrm{pr}_W^*\,\mathcal{O}_W(1) = H^0(W, \mathcal{O}_W(1)) \otimes \mathcal{O}_{\mathbb{P}^1}$ and its higher direct images vanish, by Kodaira vanishing on the Fano fourfold W. The generic fibre F is a smooth index-one Fano threefold with $(-K_F)^3 = 18$, $h^{1,2}(F) = 2$ and hence $e(F) = 0$.

By the Lefschetz pencil theorem [4], a general pencil of hyperplanes on the smooth fourfold W has a reduced finite discriminant; each singular member has precisely one ordinary quadratic singularity and the singular points lie outside the smooth base locus S. The relative differential is the section

$$\mathrm{d}p \in H^0(X, \Omega^1_X \otimes p^*T_{\mathbb{P}^1}), \qquad \mathrm{Crit}(p) = Z(\mathrm{d}p).$$

Its zero scheme has expected codimension four and, for the general pencil, consists of reduced points. Since $p^*T_{\mathbb{P}^1} = \mathcal{O}_X(2A)$ and $A^2 = 0$, the top Chern-class formula gives

$$\begin{aligned} [\mathrm{Crit}(p)] &= c_4(\Omega^1_X \otimes \mathcal{O}_X(2A)) \\ &= c_4(T_X) - 2A\,c_3(T_X) \quad \in A^4(X), \\ \deg \mathrm{Crit}(p) &= e(X) - 2\int_X A\,c_3(T_X) = e(X) - 2e(F). \end{aligned}$$

From the Hodge numbers of X recorded above, $e(X) = 1 + 2 + (22 + 2) + 2 + 1 = 30$. Thus

$$\deg \Delta = \deg \mathrm{Crit}(p) = 30.$$

The equality between the two degrees uses that the general Lefschetz pencil has one node per singular fibre. The base S is regular, so no additional critical points arise from the blow-up construction. Finally, A and H are nef and linearly independent, and their morphisms p and π_1 contract independent curve classes. Because $\rho(X) = 2$, both classes span boundary rays of the entire nef cone. The cone is therefore simplicial, and p is an elementary extremal Mori fibre space. The rationality statement follows from [2, Theorem 3.3], as in [1].

◇

References

[1] M. Bernardara, E. Fatighenti, L. Manivel and F. Tanturri, *Fano fourfolds of K3 type*, Ann. Inst. Fourier (2026), online first, 121 pp., doi:10.5802/aif.3761.

[2] A. Kuznetsov and Y. Prokhorov, *Rationality of Mukai varieties over non-closed fields*, in *Rationality of Varieties*, Progr. Math. **342**, Birkhäuser, Cham (2021), 249–290, doi:10.1007/978-3-030-75421-1_10.

[3] S. Mukai, *Fano 3-folds*, in *Complex Projective Geometry*, London Math. Soc. Lecture Note Ser. **179**, Cambridge Univ. Press (1992), 255–263.

[4] P. Deligne and N. Katz (eds.), *Groupes de monodromie en géométrie algébrique, II (SGA 7 II)*, Lecture Notes in Math. **340**, Springer, Berlin (1973), Exposé XVII.