Fano 3-104-28-b. $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathrm{Gr}(2,4),\ \mathcal{O}_{\mathrm{Gr}}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathrm{Gr}}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)\big)$.

Invariants. $h^0(-K) = 28$, $(-K)^4 = 104$, $h^{1,1} = 3$, $h^{2,1} = 2$, $h^{3,1} = 0$, $h^{2,2} = 12$, $-\chi(T) = 14$.

Description.

(1) $\pi_1 \colon X \to \mathbb{P}^2 \times \mathrm{Gr}(2,4)$ is an extremal divisorial contraction onto a fourfold of the family 2-128-33-a, the blow-up of it along a quartic del Pezzo surface.
(2) $\pi_2 \colon X \to \mathbb{P}^1 \times \mathrm{Gr}(2,4)$ is an extremal divisorial contraction onto $\mathbb{P}^1 \times Y$, where Y is a smooth complete intersection of two quadrics in $\mathbb{P}^5$: the blow-up of $\mathbb{P}^1 \times Y$ along a rational elliptic surface.
(3) $\pi_3 \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ is an extremal contraction of fiber type onto a surface $\mathbb{F}_1$, a fibration in quartic del Pezzo surfaces, the hyperplane sections of Y containing a fixed plane of $\mathbb{P}^5$.
(4) $\pi_{12} \colon X \to \mathrm{Gr}(2,4)$ maps onto Y; outside four points of Y, over which the fiber is a surface $\mathbb{F}_1$, it is a conic bundle whose discriminant is a hyperplane section of Y, a quartic del Pezzo surface.

- f, h, H denote the pullbacks of the hyperplane classes of $\mathbb{P}^1$ and $\mathbb{P}^2$ and of the Plücker class.
- $\mathrm{Nef}(X) = \langle f, h, H\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- With C the genus 2 curve associated with Y, from π_2, [BO] and [Orl]:

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(C), \mathrm{D}^{\mathrm{b}}(C), E_1, \dots, E_{16}\rangle,$$

  with $E_1, \dots, E_{16}$ exceptional objects.

Explanation.

Write $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ as a smooth quadric. The section of $\mathcal{O}_{\mathrm{Gr}}(2)$ cuts it with a second quadric in the smooth threefold Y. The section of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ on $\mathbb{P}^2 \times \mathrm{Gr}(2,4)$ is a net of hyperplanes H_m of $\mathbb{P}^5$, $m \in \mathbb{P}^2$, with base locus a plane $\Pi \subset \mathbb{P}^5$; it imposes $y \in H_m$. The section of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ on $\mathbb{P}^1 \times \mathbb{P}^2$ is a 2×3 matrix of rank 2, with kernel a point $m_0 \in \mathbb{P}^2$; it imposes that m lies on a line ℓ_t of the pencil of lines through m_0, parametrized by $t \in \mathbb{P}^1$. Its zero locus $S = \{(t,m) : m \in \ell_t\}$ is $\mathbb{P}^2$ blown up at m_0, that is $\mathbb{F}_1$. Thus

$$X = \{(t, m, y) :\ m \in \ell_t,\ y \in Y \cap H_m\}.$$

Let $D_0 = Y \cap H_{m_0}$, a quartic del Pezzo surface, and note that $Y \cap \Pi$ consists of four points of D_0.

Description of π_1. The image of π_1 is

$$Z = \{(m, y) : y \in Y \cap H_m\} = \mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2,4), \mathcal{O}_{\mathrm{Gr}}(2) \oplus \mathcal{O}(1) \boxtimes \mathcal{O}(1)\big),$$

the model of 2-128-33-a. The remaining section is $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes h$ on $\mathbb{P}^1 \times Z$, with h globally generated, so [BFMT, Lemma 3.1] gives $X \simeq \mathrm{Bl}_{D_0} Z$, where $D_0 \times \{m_0\}$ is the zero locus of the two components of the section, that is the fiber of Z over m_0. The fibers $c_f = \mathbb{P}^1 \times \{(m_0, y)\}$ have $f \cdot c_f = 1$ and $h \cdot c_f = H \cdot c_f = 0$.

Description of π_2. Over $(t, y) \in \mathbb{P}^1 \times Y$, the condition $y \in H_m$ is a line $L_y \subset \mathbb{P}^2$, or all of $\mathbb{P}^2$ if $y \in \Pi$, and the fiber of π_2 is $\ell_t \cap L_y$. It is a point, except for a line over the surface

$$T = \{(t, y) : \ L_y = \ell_t \text{ or } y \in \Pi\}.$$

As $L_y \ni m_0$ means $y \in D_0$, and then $L_y = \ell_t$ for a single t, the surface T maps isomorphically onto D_0 outside the four points of $Y \cap \Pi$, over which it contains a whole $\mathbb{P}^1$; so T is D_0 blown up at four points, a rational elliptic surface. The fibers c_h are lines of $\mathbb{P}^2$, with $h \cdot c_h = 1$ and $f \cdot c_h = H \cdot c_h = 0$, the nef class $f + H$ vanishes exactly on them, and [And, Thm. 2.3] gives $X \simeq \mathrm{Bl}_T(\mathbb{P}^1 \times Y)$, since X is Fano.

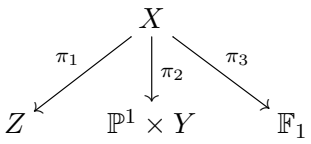
Description of π_3. The image of π_3 is $S \simeq \mathbb{F}_1$, and the fiber over (t, m) is the hyperplane section $Y \cap H_m$, which contains $Y \cap \Pi$ and is irreducible and reduced since $\mathrm{Pic}\, Y = \mathbb{Z}H$: a quartic del Pezzo surface. A line c_H in a fiber has $H \cdot c_H = 1$ and $f \cdot c_H = h \cdot c_H = 0$, and the nef class $f + h$ vanishes exactly on the curves of the fibers.

Description of π_{12}. The fiber of π_{12} over $y \in Y$ is the preimage in $S = \mathrm{Bl}_{m_0} \mathbb{P}^2$ of the line L_y: a smooth rational curve if $m_0 \notin L_y$; the union of the strict transform of L_y and of the exceptional curve if $m_0 \in L_y \neq \mathbb{P}^2$, that is $y \in D_0$ outside the four points; and the whole surface S if $y \in Y \cap \Pi$. It contracts the face $\langle c_f, c_h \rangle$ of $\overline{\mathrm{NE}}(X)$.

Cones and Hodge numbers. The classes f, h, H are nef, and the curves c_f, c_h, c_H form the dual basis:

	c_f	c_h	c_H
f	1	0	0
h	0	1	0
H	0	0	1

Hence $\mathrm{Nef}(X) = \langle f, h, H \rangle$ and $\overline{\mathrm{NE}}(X) = \langle c_f, c_h, c_H \rangle$ are dual simplicial cones, and π_1, π_2, π_3 are the three extremal contractions.



The threefold Y has $h^{2,1} = 2$, its intermediate Jacobian being the Jacobian of the genus 2 curve C associated with the pencil of quadrics, and T is rational with $h^{1,1}(T) = 10$; so $X = \mathrm{Bl}_T(\mathbb{P}^1 \times Y)$ has the Hodge numbers above. Since Y is rational, so is X. Finally, $\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathrm{D}^{\mathrm{b}}(C), \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle$ [BO], so $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1 \times Y)$ consists of two copies of $\mathrm{D}^{\mathrm{b}}(C)$ and 4 exceptional objects, and [Orl] adds $\mathrm{D}^{\mathrm{b}}(T)$, which has a full exceptional collection of length 12.

◇

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