Fano 3-135-34-a. $\mathscr{Z}\big(\mathbb{P}^1\times\mathbb{P}^2\times\mathrm{Gr}(2,5),\mathcal{O}(0,0,1)^{\oplus 2}\oplus\mathcal{Q}\boxtimes\mathcal{O}_{\mathrm{Gr}}(1)\oplus\mathcal{O}(1,1,0)\big)$, where $\mathcal{Q}$ is the rank-two universal quotient bundle on $\mathbb{P}^2$.

Invariants. $h^0(-K)=34,\quad (-K)^4=135,\quad h^{1,1}=3,\quad h^{2,1}=1,\quad h^{3,1}=0,\quad h^{2,2}=9,\ -\chi(T)=9.$

Description. Let $W\subset\mathrm{Gr}(2,5)$ be a smooth codimension-two Plücker linear section, $T\subset W$ a smooth intersection of two hyperplanes, and $C\subset T$ a smooth hyperplane section. Thus $T\simeq \mathrm{dP}_5$ and C is an elliptic quintic. Set $X_0=\mathrm{Bl}_C W$, $Y=\mathrm{Bl}_T W$ and $F=\mathrm{Bl}_p\mathbb{P}^2\simeq\mathbb{F}_1$. For the ordered factors $\mathbb{P}^1,\mathbb{P}^2,\mathrm{Gr}(2,5)$, π_i omits the ith factor.

(1) $\pi_1: X\to X_0$ is the blow-up of a surface $S\simeq T$; X_0 is Fano 2-165-40-a.
(2) $\pi_2: X\to Y$ is the blow-up of $Z\simeq C\times\mathbb{P}^1$; Y is Fano 2-240-54-a.
(3) $\pi_3: X\to F$ is a Mori fiber space with general fiber dP_5.

The fourfold X is rational.

Semiorthogonal decompositions. We have

$$\mathrm{D}^{\mathrm{b}}(X)=\langle \mathrm{D}^{\mathrm{b}}(T),\mathrm{D}^{\mathrm{b}}(C),\mathrm{D}^{\mathrm{b}}(C),\mathrm{D}^{\mathrm{b}}(W)\rangle,$$

up to the usual fully faithful functors and twists. Indeed, Orlov's blow-up formula [3], together with the full exceptional collections of lengths six and seven on W and T, respectively [1, Appendix B], gives two copies of $\mathrm{D}^{\mathrm{b}}(C)$ and 13 exceptional objects.

Explanation.

On $\mathbb{P}^2$ the universal sequence is

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^2}(-1)\longrightarrow\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}\longrightarrow\mathcal{Q}\longrightarrow 0. \tag{1}$$

The two Plücker equations define W, a smooth index-three fourfold of degree five with $\mathrm{Pic}(W)=\mathbb{Z}H$. The remaining sections give

$$X=\mathscr{Z}\big(\mathbb{P}^1\times\mathbb{P}^2\times W,\mathcal{O}(1,1,0)\oplus\mathcal{Q}\boxtimes\mathcal{O}_W(1)\big).$$

A general $(1,1)$ divisor in $\mathbb{P}^1\times\mathbb{P}^2$ is $F=\mathrm{Bl}_p\mathbb{P}^2$. Therefore

$$X=\mathscr{Z}\big(F\times W,\mathcal{Q}|_F\boxtimes\mathcal{O}_W(1)\big),\qquad -K_X=L+h+H,$$

where the anticanonical class is the restriction of the ample bundle $\mathcal{O}(1,1,1)$. A general defining section gives a smooth Fano fourfold.

Description of π_1. For $u:\mathbb{P}^2\times W\to W$, the projective-space cohomology and (1) give

$$u_*(\mathcal{Q}\boxtimes\mathcal{O}_W(1))=\mathcal{O}_W(1)^{\oplus 3},\qquad R^iu_*(\mathcal{Q}\boxtimes\mathcal{O}_W(1))=0\ (i>0).$$

The arbitrary section is consequently a general triple $s_0,s_1,s_2\in H^0(W,\mathcal{O}_W(1))$, inducing

$$\psi_3:\mathcal{O}_W(-1)\xrightarrow{(s_0,s_1,s_2)}\mathcal{O}_W^{\oplus 3}.$$

The rank-zero locus $C=D_0(\psi_3)$ has expected codimension three and is smooth and connected by the Bertini Theorem. Its Chern class, normal bundle, and canonical divisor are

$$[C]=c_3(\mathcal{O}_W(1)^{\oplus 3})=H^3,\quad N_{C/W}=\mathcal{O}_C(1)^{\oplus 3},\quad K_C=0.$$

Since $H^4=5$, C is an elliptic quintic. The section on $\mathbb{P}^2\times W$ vanishes along the equations

$$\mathrm{rank}\begin{pmatrix}s_0 & s_1 & s_2\\ z_0 & z_1 & z_2\end{pmatrix}\le 1.$$

They define the closure of the graph of $W \dashrightarrow \mathbb{P}^2$, $w \mapsto [s_0(w) : s_1(w) : s_2(w)]$. As C is a regular complete intersection, the rank-two Cayley trick [1, Section 3] yields

$$X_0 = \mathscr{Z}(\mathbb{P}^2 \times W, \mathcal{Q} \boxtimes \mathcal{O}_W(1)) \simeq \mathrm{Bl}_C W.$$

Write $p = [0 : 0 : 1]$; then

$$F = \{([a : b], [z_0 : z_1 : z_2]) : az_1 - bz_0 = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^2.$$

The map $X \to X_0$ is the base change of $F \to \mathbb{P}^2$. The zero scheme of

$$\sigma_S : \mathcal{O}_{X_0} \xrightarrow{(z_0,z_1)} \mathcal{O}_{X_0}(1,0)^{\oplus 2}$$

is $S = (X_0 \to \mathbb{P}^2)^{-1}(p)$. It maps isomorphically to $T = D_0(\psi_2) \subset W$, where

$$\psi_2 : \mathcal{O}_W(-1) \xrightarrow{(s_0,s_1)} \mathcal{O}_W^{\oplus 2}.$$

The general pair is regular, so T is smooth and connected, with

$$[T] = c_2(\mathcal{O}_W(1)^{\oplus 2}) = H^2, \quad N_{T/W} = \mathcal{O}_T(1)^{\oplus 2}, \quad -K_T = H|_T,$$

and thus $T \simeq \mathrm{dP}_5$. Likewise σ_S is regular,

$$[S] = c_2(\mathcal{O}_{X_0}(1,0)^{\oplus 2}) = h^2, \qquad N_{S/X_0} = \mathcal{O}_S^{\oplus 2}.$$

The regular base change of $\mathrm{Bl}_p \mathbb{P}^2$ along $X_0 \to \mathbb{P}^2$ gives $X \simeq \mathrm{Bl}_S X_0$, with exceptional divisor $E_1 = h - L$.

Description of π_2. Let $B = \mathbb{P}^1 \times W$. The incidence $F \times W$ is the projectivization of the rank-two kernel in

$$0 \longrightarrow \mathcal{E} = \mathcal{O}_B(-1,0) \oplus \mathcal{O}_B \longrightarrow \mathcal{O}_B^{\oplus 3} \xrightarrow{(-b,a,0)} \mathcal{O}_B(1,0) \longrightarrow 0,$$

with $[a : b] \in \mathbb{P}^1$. Compose the pullback of ψ_3 with the indicated quotient:

$$\mathcal{O}_B(0,-1) \xrightarrow{(s_0,s_1,s_2)} \mathcal{O}_B^{\oplus 3} \xrightarrow{(-b,a,0)} \mathcal{O}_B(1,0).$$

The composite is $as_1 - bs_0 \in H^0(B, \mathcal{O}_B(1,1))$. Consequently, the scheme-theoretic image of π_2 is

$$Y = \mathscr{Z}_B(as_1 - bs_0), \qquad \mathrm{pr}_{W*}\,\mathcal{O}_B(1,1) = \mathcal{O}_W(1)^{\oplus 2}, \quad R^i \mathrm{pr}_{W*}\,\mathcal{O}_B(1,1) = 0 \ (i > 0).$$

For each $([a : b], w) \in Y$ the section $(s_0, s_1, s_2)(w)$ lies in $\mathcal{E}_{([a:b],w)}$, so the relative equation admits a projective solution; this is also the graph of $W \dashrightarrow \mathbb{P}^1$, $[s_0 : s_1]$. The two sections are general, and their base scheme is the smooth surface T. Thus $Y \simeq \mathrm{Bl}_T W$ by the graph construction, with exceptional divisor $E_T = H - L$.

On Y the triple (s_0, s_1, s_2) lifts to $\mathcal{E} \otimes \mathcal{O}_Y(0,1)$. In terms of the decomposition $\mathcal{E} = \mathcal{O}_Y(-1,0) \oplus \mathcal{O}_Y$, its components define

$$\vartheta : \mathcal{O}_Y \longrightarrow \mathcal{O}_Y(-1,1) \oplus \mathcal{O}_Y(0,1), \qquad \vartheta = (\lambda, s_2),$$

where $s_0 = a\lambda$, $s_1 = b\lambda$. Since s_2 is independent and general after s_0, s_1, its vanishing cuts the exceptional divisor $\{\lambda = 0\} \simeq T \times \mathbb{P}^1$ transversely in

$$Z = D_0(\vartheta) = \{\lambda = s_2 = 0\} \simeq C \times \mathbb{P}^1.$$

This locus is smooth of codimension two, with

$$[Z] = c_2(\mathcal{O}_Y(-1,1) \oplus \mathcal{O}_Y(0,1)) = H(H - L),$$
$$N_{Z/Y} = \mathcal{O}_Z(-1,1) \oplus \mathcal{O}_Z(0,1), \qquad c(N_{Z/Y}) = (1 + H - L)(1 + H)|_Z.$$

Let $\varrho : \mathbb{P}_Y(\mathcal{E}|_Y) \to Y$ parametrize lines in $\mathcal{E}|_Y$, with universal line $\mathcal{O}(-1) \subset \varrho^*\mathcal{E}$. The quotient $\varrho^*\mathcal{E}/\mathcal{O}(-1)$ is the line bundle $\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1) \otimes \varrho^*\mathcal{O}_Y(-1,0)$. The relative equation of X is consequently a section of

$$\mathcal{O}_{\mathbb{P}_Y(\mathcal{E})}(1) \otimes \varrho^*\mathcal{O}_Y(-1,1),$$

whose direct image, with no higher direct images, is

$$\varrho_*(\mathcal{O}(1) \otimes \varrho^*\mathcal{O}_Y(-1,1)) = \mathcal{E}^\vee|_Y \otimes \mathcal{O}_Y(-1,1) = \mathcal{O}_Y(0,1) \oplus \mathcal{O}_Y(-1,1).$$

Its pushed-forward section is ϑ. The rank-two Cayley trick [1, Section 3], applied to this regular section, identifies $X \simeq \mathrm{Bl}_Z Y$, with $E_2 = H - h$.

Description of π_3. Let $v : F \times W \to F$. Then

$$v_*(\mathcal{Q}|_F \boxtimes \mathcal{O}_W(1)) = \mathcal{Q}|_F \otimes H^0(W, \mathcal{O}_W(1)), \qquad R^i v_*(\mathcal{Q}|_F \boxtimes \mathcal{O}_W(1)) = 0 \ (i > 0).$$

The section induces

$$\Phi : (\mathcal{Q}|_F)^\vee \longrightarrow H^0(W, \mathcal{O}_W(1)) \otimes \mathcal{O}_F.$$

The independence of s_0, s_1, s_2 makes Φ of constant rank two. The corresponding pair of hyperplanes cuts out each fiber of π_3 in W; a general fiber is a smooth dP_5. All fibers are connected by connectedness of ample hyperplane sections of W. The relative Jacobian is the map of ranks $4 \to 2$

$$\delta : \mathrm{pr}_W^* T_W|_X \longrightarrow \mathrm{pr}_F^*(\mathcal{Q}|_F) \otimes \mathcal{O}_X(0,0,1).$$

The critical scheme is $D_1(\delta)$. Since δ is a constrained differential, not a general homomorphism, no claim of expected codimension is made. If it has codimension three, Thom–Porteous [2] gives

$$[D_1(\delta)] = \det\big(c_{1+j-i}(\mathrm{pr}_F^*\,\mathcal{Q} \otimes \mathcal{O}_X(0,0,1) - \mathrm{pr}_W^*\,T_W)\big)_{1 \le i,j \le 3}.$$

As $\rho(X) = 3$ and $\rho(F) = 2$, π_3 has relative Picard number one and is an elementary K-negative fiber-type contraction.

The nef and Mori cones. The three classes L, h, H are clearly nef. For a line f_1 in an exceptional fiber of π_1, a line f_2 in an exceptional fiber of π_2, and a line f_3 in a general dP_5 fiber of π_3, we have the intersection numbers

	L	h	H
f_1	1	0	0
f_2	0	1	0
f_3	0	0	1

.

Since $\rho(X) = 3$, these classes form dual bases of $N_1(X)$ and $N^1(X)$. A nef divisor has nonnegative degree on every f_i, while every nonnegative combination of L, h, H is nef. Thus

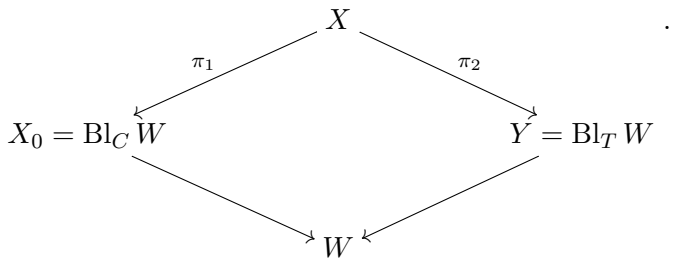
$$\mathrm{Nef}(X) = \mathbb{R}_{\ge 0}L + \mathbb{R}_{\ge 0}h + \mathbb{R}_{\ge 0}H$$

is simplicial. By duality, the Mori cone is

$$\overline{\mathrm{NE}}(X) = \mathbb{R}_{\ge 0}[f_1] + \mathbb{R}_{\ge 0}[f_2] + \mathbb{R}_{\ge 0}[f_3].$$

In particular, it is simplicial, and the three projections are exactly its elementary extremal contractions.

Observation. The two divisorial contractions fit into the commutative diagram



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Note that since W is rational, so is X.

◇

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