Fano 3-142-35-a. $\mathscr{Z}\big(\mathrm{Fl}(1,3,4)\times\mathbb{P}^3,\ \mathcal{U}_1^\vee\oplus(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee\boxtimes\mathcal{O}(1)\oplus\det\mathcal{U}_3^\vee\boxtimes\mathcal{O}(1)\big)$.

Invariants. $h^0(-K)=35,\ (-K)^4=142,\ h^{1,1}=3,\ h^{2,1}=0,\ h^{3,1}=0,\ h^{2,2}=13,\ -\chi(T)=10.$

Description. Let X be the zero locus of a general section of the indicated bundle. Its three extremal divisorial contractions are:

(1) $\pi_1\colon X\to W$ is the blow-up of a Fano fourfold of type 2-320-69-d along a smooth cubic surface T.
(2) $\pi_2\colon X\to Z$ is the blow-up of a Fano fourfold of type 2-256-57-a along a dP_7.
(3) $\pi_3\colon X\to F$ is the blow-up of a Fano fourfold of type 2-432-90-d along a smooth rational surface S with $K_S^2=0$.

The two-dimensional faces of $\overline{\mathrm{NE}}(X)$ give a dP_7-fibration and two conic bundles. In particular, X is rational.

Semiorthogonal decompositions. Let $\pi=\pi_3\colon X=\mathrm{Bl}_S F\to F$, with exceptional divisor E, inclusion $i\colon E\hookrightarrow X$, and projection $q\colon E\to S$. Since S is smooth of codimension two in F, Orlov's blow-up formula [Orl92] gives

$$\mathrm{D}^{\mathrm{b}}(X)=\langle i_*(q^*\,\mathrm{D}^{\mathrm{b}}(S)\otimes\mathcal{O}_E(-1)),\ \pi^*\,\mathrm{D}^{\mathrm{b}}(F)\rangle,$$

where $\mathcal{O}_E(-1)=\mathcal{O}_E(E)$. The projection $F\to\mathbb{P}(\Lambda)\simeq\mathbb{P}^2$ is a $\mathbb{P}^2$-bundle with relative hyperplane class L. The projective bundle formula and Beilinson's collection give a full exceptional collection on F consisting of the nine line bundles $\mathcal{O}_F(ja+kL)$, for $0\le j,k\le 2$. Moreover, S is a smooth rational surface with $K_S^2=0$, hence admits a full exceptional collection $(\mathcal{E}_1,\dots,\mathcal{E}_{12})$, by the standard collections on minimal rational surfaces and Orlov's blow-up formula. Consequently, X has a full exceptional collection of length 21:

$$\begin{aligned}\big\langle & i_*(q^*\mathcal{E}_1\otimes\mathcal{O}_E(-1)),\dots,i_*(q^*\mathcal{E}_{12}\otimes\mathcal{O}_E(-1)),\\ & \mathcal{O}_X,\mathcal{O}_X(a),\mathcal{O}_X(2a),\\ & \mathcal{O}_X(L),\mathcal{O}_X(a+L),\mathcal{O}_X(2a+L),\\ & \mathcal{O}_X(2L),\mathcal{O}_X(a+2L),\mathcal{O}_X(2a+2L)\big\rangle.\end{aligned}$$

Explanation.

Setup. We use the convention that $\mathbb{P}(\mathcal{E})$ parametrizes lines in $\mathcal{E}$, and regard $\mathrm{Fl}(1,3,V_4)$ as a subvariety of $\mathbb{P}(V_4)\times\mathrm{Gr}(3,V_4)$. Write $M_1=\mathbb{P}(V_4)$, $M_2=\mathrm{Gr}(3,V_4)$, $M_3=\mathbb{P}^3$, and let $a=\mathcal{U}_1^\vee$, $L=\det\mathcal{U}_3^\vee$, $h=\mathcal{O}_{M_3}(1)$ denote the corresponding hyperplane classes. The first section fixes a three-dimensional subspace $\Lambda\subset V_4$; the second is a general web $y\mapsto\omega_y\in\bigwedge^2 V_4^\vee$; the third identifies M_3 with $\mathbb{P}(V_4)$. Thus

$$X=\{(u,U_3,y):\ u\in\mathbb{P}(\Lambda),\ u,y\in U_3,\ \omega_y(u,U_3)=0\}.$$

By Borel–Weil, the summands have sections $V_4^\vee$, $V_4^\vee\otimes\bigwedge^2 V_4^\vee$ and $V_4^\vee\otimes V_4=\mathrm{Hom}(V_4,V_4)$. The last section is an isomorphism, which we use to identify M_3 with $\mathbb{P}(V_4)$. All summands are globally generated, so X is smooth of dimension $5+3-4=4$. We have $-K_{\mathrm{Fl}(1,3,4)}=3a+3L$, $-K_{M_3}=4h$, and the summands have first Chern classes a, $(L-a)+2a+2h$ and $L+h$. Hence $-K_X=a+L+h$.

The web. A general web is a general $\mathbb{P}^3\subset\mathbb{P}(\bigwedge^2 V_4^\vee)$. It meets the Klein quadric in a smooth quadric surface Q_{Pf} of decomposable forms, whose kernels form the linear congruence of lines meeting two skew lines ℓ_1,ℓ_2. Exactly one line of this congruence lies in the plane $\mathbb{P}(\Lambda)$, namely the line through $\ell_1\cap\mathbb{P}(\Lambda)$ and $\ell_2\cap\mathbb{P}(\Lambda)$.

Description of π_1. The projection π_1 forgets u; it is supported by $L+h$. Fix (U_3, y) with $y \in U_3$. Then $\omega_y|_{U_3}$ is a 2-form on a 3-space, hence has a kernel line $\kappa(U_3, y)$, or vanishes. The condition $\omega_y(u, U_3) = 0$ means $u \subset \ker(\omega_y|_{U_3})$. Since $\bigwedge^2 U_3^\vee \simeq U_3 \otimes \det U_3^\vee$, κ is a section of $\mathcal{U}_3 \otimes \mathcal{O}(L+h)$, and $\lambda(\kappa)$ is a section of $\mathcal{O}(L+h)$ on the incidence variety $\{y \in U_3\}$, a divisor of bidegree $(1,1)$ in $\mathbb{P}^{3\vee} \times \mathbb{P}^3$. Hence

$$\pi_1(X) = W = \{y \in U_3\} \cap \{\lambda(\kappa) = 0\},$$

a complete intersection of two $(1,1)$-divisors, with $-K_W = 2L + 2h$.

The fiber of π_1 is a point unless $\omega_y|_{U_3} = 0$, in which case it is the line $\mathbb{P}(U_3 \cap \Lambda)$. Now $\omega_y|_{U_3} = 0$ forces ω_y to be degenerate and $U_3 \supset K_y$, so $U_3 = K_y + y$ when $y \notin K_y$. This gives the surface T, mapping birationally onto Q_{Pf} and containing a line over each of the finitely many $y \in Q_{\mathrm{Pf}} \cap K_y$. The description as the blow-up along the smooth center T of codimension two follows from the Cayley trick for a rank-two bundle [IM15, 3.7]: locally along T, the equation linear in u is the incidence equation for two generators of the ideal of T. Thus $X = \mathrm{Bl}_T W$, with $E_1 = L + h - a$. The blow up formulas give

$$(h^2)_T = 2, \quad (Lh)_T = 4, \quad (L^2)_T = 3, \quad c_1(N_T) \cdot h = 8, \quad c_1(N_T) \cdot L = 11.$$

With $c_1(N_T) = (2L + 2h)|_T + K_T$, these give $K_T \cdot h = -4$ and $K_T \cdot L = -3$. Writing $T = \mathrm{Bl}_N(\mathbb{P}^1 \times \mathbb{P}^1)$ with $h|_T = (1,1)$ and $L|_T = (p,q) - \sum E_i$, we find $N = 5$ and $L|_T = (2,2) - \sum_{i=1}^5 E_i = -K_T$. So T is a smooth cubic surface, embedded by U_3 as a cubic surface in $\mathbb{P}^{3\vee}$. Moreover $c_2(N_T) = c_1^2 + E_1^4 = 27 - 18 = 9$. The curve c_a satisfies $a \cdot c_a = 1$, $L \cdot c_a = h \cdot c_a = 0$ and $-K \cdot c_a = 1$.

Description of π_2. The projection π_2 forgets U_3; it is supported by $a + h$. Given (u, y), a point of the fiber is a hyperplane $U_3 \ni u, y$ contained in $\ker \omega_y(u, \cdot)$.

- If $\omega_y(u, \cdot) \neq 0$, then $U_3 = \ker \omega_y(u, \cdot)$, which contains u and must contain y: $\omega_y(u, y) = 0$. So $\pi_2(X) = Z$, a divisor of bidegree $(1,2)$ in $\mathbb{P}(\Lambda) \times \mathbb{P}^3$, with $-K_Z = 2a + 2h$, $(-K_Z)^4 = 256$, $h^0(-K_Z) = 57$ and $\chi_{\mathrm{top}}(Z) = 16$.
- If $\omega_y(u, \cdot) = 0$, i.e. $y \in Q_{\mathrm{Pf}}$ and $u \in K_y \cap \Lambda$, the fiber is the line of planes through $\langle u, y \rangle$.

This locus is S', which maps isomorphically onto Q_{Pf}, except over the single y with $K_y \subset \mathbb{P}(\Lambda)$, where it contains the line K_y. Locally along S', the equations defining the hyperplane U_3 reduce to the incidence equation for two generators of the ideal of S'. The same Cayley trick for a rank-two bundle [KKLL17, 3.2] gives $X = \mathrm{Bl}_{S'} Z$, with $E_2 = a + h - L$. The blow up formulas give $(a^2)_{S'} = 1$, $(ah)_{S'} = 2$ and $(h^2)_{S'} = 2$. Hence $S' \simeq \mathrm{Bl}_1(\mathbb{P}^1 \times \mathbb{P}^1)$ with $h = (1,1)$ and $a = (1,1) - E$, which maps birationally onto $\mathbb{P}(\Lambda)$. The curve c_L satisfies $L \cdot c_L = 1$, $a \cdot c_L = h \cdot c_L = 0$ and $-K \cdot c_L = 1$.

Description of π_3. The projection π_3 forgets y; it is supported by $a + L$, and its image is F, the fourfold of type 2-432-90-d. After the last section identifies y with a vector in U_3, the remaining equations are linear in y and give

$$\varphi \colon \mathcal{U}_3 \longrightarrow (\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee, \qquad y \mapsto \omega_y(u, \cdot)|_{U_3},$$

from rank 3 to rank 2, and $X = \mathbb{P}(\ker\varphi) \subset \mathbb{P}_F(\mathcal{U}_3)$. The fiber is a point, a line or a plane according as $\operatorname{rank}\varphi = 2, 1, 0$. By generality $D_0(\varphi)$, of expected codimension 6, is empty, and $S = D_1(\varphi)$ is a smooth surface. Since $D_0(\varphi) = \varnothing$, near each point of S one can split off a rank-one isomorphism from φ. The remaining map is a row of two functions cutting out S scheme-theoretically, and $\mathbb{P}(\ker\varphi)$ is locally their Cayley incidence variety. By the Cayley trick for a rank-two bundle [IM15, 3.7], $X = \operatorname{Bl}_S F$, with $E_3 = a + 2L - h$. The blow up formulas give

$$(a^2)_S = 2, \quad (aL)_S = 5, \quad (L^2)_S = 3, \quad K_S \cdot a = -2, \quad K_S \cdot L = -3,$$

and $b_2(S) = h^{2,2}(X) - h^{2,2}(F) = 10$. Hence S is rational, with $K_S^2 = 0$ and $\chi_{\text{top}}(S) = 12$; as a check, $c_2(N_S) = c_1(N_S)^2 + E_3^4 = 69 - 48 = 21$.

The map $u\colon S \to \mathbb{P}(\Lambda)$ has degree 2. For fixed u, write $U_3 = \ker f$. The conditions are a linear and a quadratic condition on $f \in \mathbb{P}((V_4/u)^\vee)$, so there are two solutions in general. Over the two points of $\mathbb{P}(\Lambda)$ where $y \mapsto \omega_y(u, \cdot)$ drops rank, the fiber is a curve; their number is $c_2(\Omega_{\mathbb{P}^3}(2)|_{\mathbb{P}^2}) = 2$. The curve c_h satisfies $h \cdot c_h = 1$, $a \cdot c_h = L \cdot c_h = 0$ and $-K \cdot c_h = 1$.

Extremal contractions and cones. The curves c_a, c_L, c_h are dual to the classes a, L, h. Therefore

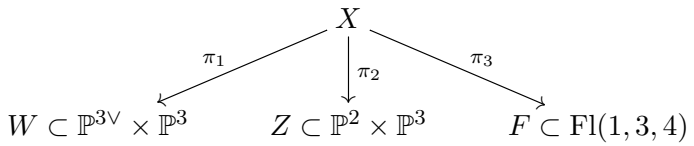
$$\overline{\mathrm{NE}}(X) = \langle c_a, c_L, c_h\rangle, \qquad \mathrm{Nef}(X) = \langle a, L, h\rangle,$$

both simplicial, with

	c_a	c_L	c_h
a	1	0	0
L	0	1	0
h	0	0	1
E_1	-1	1	1
E_2	1	-1	1
E_3	1	2	-1
$-K$	1	1	1

The two-dimensional faces give the following contractions.

- $a^\perp = \langle c_L, c_h\rangle\colon X \to \mathbb{P}(\Lambda)$. The fiber over u is the quadric $\{y : \omega_y(u, y) = 0\}$ blown up at the point $y_0(u)$ where $\omega_y(u, \cdot) = 0$, i.e. a del Pezzo surface of degree 7.
- $L^\perp = \langle c_a, c_h\rangle\colon X \to \mathbb{P}^{3\vee}$. Over U_3, $\kappa(U_3, y)$ is linear in $y \in \mathbb{P}(U_3)$, and $\kappa \in \Lambda$ cuts a line. So the fiber is a conic, with $-K$-degree 2.
- $h^\perp = \langle c_a, c_L\rangle\colon X \to M_3$. For ω_y symplectic, $U_3 = u^{\perp\omega_y}$, and $\omega_y(u, y) = 0$ cuts a line in $\mathbb{P}(\Lambda)$; so this is again a conic bundle.



◇

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