Fano 3-142-35-b. $\mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^4_x \times \mathbb{P}^4_y,\ \mathcal{O}_{\mathbb{P}^4_x}(1) \boxtimes \mathcal{Q}^\vee_{\mathbb{P}^4_y}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^4_y}(1)\big)$.

Invariants. $h^0(-K) = 35,\ (-K)^4 = 142,\ h^{1,1} = 3,\ h^{2,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 9,\ -\chi(T) = 7$.

Description.

(1) $\pi_1 \colon X \to \mathbb{P}^4_x \times \mathbb{P}^4_y$ is an extremal divisorial contraction onto a fourfold of the family 2-225-51-b, the blow-up of it along a quartic del Pezzo surface.
(2) $\pi_2 \colon X \to \mathbb{P}^1 \times \mathbb{P}^4_y$ is an extremal divisorial contraction onto a fourfold of the family 2-512-105-b, the blow-up of it along an elliptic quintic scroll blown up at 5 points.
(3) $\pi_3 \colon X \to \mathbb{P}^1 \times \mathbb{P}^4_x$ is an extremal divisorial contraction onto a fourfold of the family 2-297-65-a, the blow-up of it along $\mathbb{P}^1 \times C$, with C an elliptic normal quintic curve.

- s, a, c denote the pullbacks of the hyperplane classes of the three factors.
- $\mathrm{Nef}(X) = \langle s, a, c\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_3 and [Orl]: $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(C), \mathrm{D}^{\mathrm{b}}(C), E_1, \dots, E_{13}\rangle$, with $E_1, \dots, E_{13}$ exceptional objects.

Explanation.

Write the factors as $\mathbb{P}^1$, $\mathbb{P}^4_x = \mathbb{P}(V_5)$ and $\mathbb{P}^4_y = \mathbb{P}(W_5)$. On $\mathbb{P}^4_y$ we have $H^0(\mathcal{Q}^\vee(1)) = \bigwedge^2 W_5^\vee$, and a 2-form ω vanishes as a section at y if and only if $y \in \ker\omega$. So the first section is a general linear map $x \mapsto \omega_x \in \bigwedge^2 W_5^\vee$, and imposes $y \in \ker\omega_x$; the second is a general pencil of linear forms λ_t on W_5, and imposes $\lambda_t(y) = 0$. Thus

$$X = \{(t, x, y) :\ y \in \ker\omega_x,\ \lambda_t(y) = 0\}.$$

Let $\Pi \subset \mathbb{P}^4_y$ be the base plane of the pencil λ_t. The fourfold

$$W = \{(x, y) :\ y \in \ker\omega_x\} = \mathscr{Z}\big(\mathbb{P}^4 \times \mathbb{P}^4, \mathcal{O}(1) \boxtimes \mathcal{Q}^\vee(1)\big)$$

is the model of 2-225-51-b. Its projection to $\mathbb{P}^4_x$ gives $W = \mathrm{Bl}_C \mathbb{P}^4_x$, where the curve C is a codimension-5 linear section of $\mathrm{Gr}(2, W_5^\vee) \subset \mathbb{P}(\bigwedge^2 W_5^\vee)$, that is an elliptic normal quintic curve. Its projection to $\mathbb{P}^4_y$ gives $W = \mathrm{Bl}_T \mathbb{P}^4_y$, where T is an elliptic quintic scroll.

Description of π_1. Since X is a section of $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_W(1)$ on $\mathbb{P}^1 \times W$, by [BFMT, Lemma 3.1] $X \simeq \mathrm{Bl}_{S_1} W$, where $S_1 = \{(x, y) \in W :\ y \in \Pi\}$ is the preimage of Π in $W = \mathrm{Bl}_T \mathbb{P}^4_y$. Since Π meets T transversally in 5 points, S_1 is Π blown up at these points, a quartic del Pezzo surface. A fiber f_1 of π_1 over S_1 has $s \cdot f_1 = 1$ and $a \cdot f_1 = c \cdot f_1 = 0$.

Description of π_2. The image of π_2 is $D_2 = \{(t, y) : \lambda_t(y) = 0\} = \mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^4, \mathcal{O}(1, 1)\big)$, the model of 2-512-105-b, and $X = W \times_{\mathbb{P}^4_y} D_2$. So the fiber of π_2 over (t, y) is a point if $y \notin T$, and a line if $y \in T$. The preimage of T in D_2 is $T' = \{(t, y) : y \in T,\ \lambda_t(y) = 0\}$, which is T blown up at the 5 base points of the pencil $\lambda_t|_T$, that is at $T \cap \Pi$. A line f_2 in a fiber over T' has $a \cdot f_2 = 1$ and $s \cdot f_2 = c \cdot f_2 = 0$, the nef class $s + c$ vanishes exactly on these lines, and [And, Thm. 2.3] gives $X \simeq \mathrm{Bl}_{T'} D_2$, since X is Fano.

Description of π_3. For $x \notin C$, the point y is forced to be $\ker \omega_x$, and the condition becomes $q_t(x) = 0$, where $q_t = \lambda_t(\ker \omega_x)$ is a pencil of quadrics through C. So the image of π_3 is

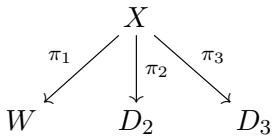
$$D_3 = \{(t, x) : \ q_t(x) = 0\} = \mathscr{Z}\big(\mathbb{P}^1 \times \mathbb{P}^4, \mathcal{O}(1,2)\big),$$

and by [BFMT, Lemma 3.1] $D_3 = \mathrm{Bl}_S \mathbb{P}^4_x$, with $S = \{q_0 = q_1 = 0\}$ a smooth quartic del Pezzo surface containing C; this is the model of 2-297-65-a. For $x \in C$ and any t, the fiber of π_3 is the line $\mathbb{P}(\ker \omega_x) \cap \{\lambda_t = 0\}$, since for a general pencil no λ_t vanishes on a plane $\mathbb{P}(\ker \omega_x)$. So the fiber of π_3 is a point outside $\mathbb{P}^1 \times C \subset D_3$ and a line over it. A line f_3 in such a fiber has $c \cdot f_3 = 1$ and $s \cdot f_3 = a \cdot f_3 = 0$, the nef class $s + a$ vanishes exactly on these lines, and [And, Thm. 2.3] gives $X \simeq \mathrm{Bl}_{\mathbb{P}^1 \times C} D_3$.

Cones and the other projections. The classes s, a, c are nef, and f_1, f_2, f_3 form the dual basis:

	f_1	f_2	f_3
s	1	0	0
a	0	1	0
c	0	0	1

Hence $\mathrm{Nef}(X) = \langle s, a, c\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2, f_3\rangle$ are dual simplicial cones, and π_1, π_2, π_3 are the three extremal contractions.



Hodge numbers and rationality. From π_3, X is obtained from $\mathbb{P}^4$ by blowing up the quartic del Pezzo surface S, with $h^{1,1}(S) = 6$, and then the surface $\mathbb{P}^1 \times C$, with $h^{1,0} = 1$ and $h^{1,1} = 2$; this gives $h^{2,1}(X) = 1$ and $h^{2,2}(X) = 1{+}6{+}2 = 9$. In particular X is rational, and [Orl] gives the semiorthogonal decomposition, with $\mathrm{D^b}(\mathbb{P}^1 \times C) = \langle \mathrm{D^b}(C), \mathrm{D^b}(C)\rangle$, 5 exceptional objects from $\mathbb{P}^4$ and 8 from S.

◇

References

[And] T. Ando, *On extremal rays of the higher dimensional varieties*, Invent. Math. 81 (1985), 347–357.

[BFMT] M. Bernardara, E. Fatighenti, L. Manivel, F. Tanturri, *Fano fourfolds of K3 type*, Ann. Inst. Fourier (2026), online first, doi:10.5802/aif.3761.

[Ful] W. Fulton, *Intersection theory*, 2nd ed., Springer, 1998.

[Orl] D. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. 41 (1993), 133–141.