Fano 3-157-38-a. $\mathscr{Z}\big(\mathrm{Fl}(1,3,5)\times\mathbb{P}^3,\ \mathcal{U}_1^\vee\boxtimes\mathcal{O}(1)\oplus(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee\boxtimes\mathcal{O}(1)\oplus(V_5/\mathcal{U}_3)^{\oplus 2}\big)$.

Invariants. $h^0(-K)=38$, $(-K)^4=157$, $h^{1,1}=3$, $h^{2,1}=0$, $h^{3,1}=0$, $h^{2,2}=14$, $-\chi(T)=11$.

Description. The three natural projections give the following extremal contractions:

(1) $\pi_1\colon X\to Z\subset\mathbb{P}^4\times\mathbb{P}^3$ is the blow-up of a smooth complete intersection Z of three divisors of bidegree $(1,1)$ along a surface $S_0\simeq\mathbb{F}_1$.
(2) $\pi_2\colon X\to D\subset\mathbb{P}^2\times\mathbb{P}^3$ is the blow-up of a smooth divisor D of bidegree $(1,2)$ along a sextic del Pezzo surface Σ.
(3) $\pi_3\colon X\to W\simeq\mathrm{Bl}_{\ell_0}\mathbb{P}^4$ is the blow-up along the strict transform $\tilde{T}$ of a surface $T\simeq\mathrm{Bl}_{10}\mathbb{P}^2$, containing the line $\ell_0\subset\mathbb{P}^4$; in particular $\tilde{T}\simeq T$.

Writing $a=\mathcal{U}_1^\vee$, $L=\det\mathcal{U}_3^\vee$ and $h=\mathcal{O}_{\mathbb{P}^3}(1)$ on X, we have $-K_X=a+L+h$, and

$$E_1=a-L,\qquad E_2=L+h-a,\qquad E_3=2a+L-h.$$

Moreover, $\mathrm{Nef}(X)=\langle a,L,h\rangle$ and $\overline{\mathrm{NE}}(X)=\langle f_1,f_2,f_3\rangle$, where f_i is contracted by π_i. The variety X is rational, as it is birational to $\mathbb{P}^4$

Semiorthogonal decompositions. From π_3: the two blow-ups $W=\mathrm{Bl}_{\ell_0}\mathbb{P}^4$ and $X=\mathrm{Bl}_{\tilde{T}}W$ give, by Orlov's formula [1], a semiorthogonal decomposition with components $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^4)$, two copies of $\mathrm{D}^{\mathrm{b}}(\ell_0)$, and $\mathrm{D}^{\mathrm{b}}(T)$. Since $\ell_0\simeq\mathbb{P}^1$ and $T\simeq\mathrm{Bl}_{10}\mathbb{P}^2$, Beilinson's theorem [2] and Orlov's formula provide full exceptional collections of lengths 5, 2, 2, and 13 on these components. Hence $\mathrm{D}^{\mathrm{b}}(X)$ admits a full exceptional collection of length $5+2+2+13=22=\chi_{\mathrm{top}}(X)$.

Explanation.

Setup. Write $M_1=\mathbb{P}(V_5)=\mathbb{P}^4$, $M_2=\mathrm{Gr}(3,V_5)$ and $M_3=\mathbb{P}(A_4)=\mathbb{P}^3$. The defining sections give a bilinear form $\beta\colon V_5\times A_4\to\mathbb{C}$, a family of two-forms $\omega\colon A_4\to\bigwedge^2 V_5^\vee$ and two vectors v_1,v_2. Set $P_2=\langle v_1,v_2\rangle$ and $\ell_0=\mathbb{P}(P_2)$. Thus

$$X=\{(u,U_3,z):\ u\in U_3\supset P_2,\ \beta(u,z)=0,\ \omega_z(u,U_3)=0\},$$

where U_3/P_2 is parametrized by $\mathbb{P}^2_y=\mathbb{P}(V_5/P_2)$. Put $H_0=\{z:\omega_z(v_1,v_2)=0\}\simeq\mathbb{P}^2$. As in `3-310-67-a`, the weight $(1,0\,|\,1,1\,|\,1)$ gives $V_5/\mathcal{U}_3$. We have

$$c_1(\mathrm{Fl}(1,3,5)\times\mathbb{P}^3)=3a+4L+4h,$$

and the summands have total first Chern class $(a+h)+(L+a+2h)+2L$, hence $-K_X=a+L+h$.

Description of π_1. The projection π_1 forgets U_3; it is supported by $a+h$. Its image is $Z = \{\beta(u,z) = \omega_z(u,v_1) = \omega_z(u,v_2) = 0\} \subset \mathbb{P}^4 \times \mathbb{P}^3$, a complete intersection of three $(1,1)$-divisors (type 2-192-45-f). For $u \notin \ell_0$ we have $U_3 = P_2 + u$, and the conditions become the three $(1,1)$-equations of Z. The numbers $a^4 = 1$, $a^3h = 3$, $a^2h^2 = 3$, $ah^3 = 1$ are those of such a complete intersection.

For $u = x_1v_1 + x_2v_2 \in \ell_0$, the space $U_3 = P_2 + y$ is free. The conditions read:

- $\beta(u,z) = 0$;
- $\omega_z(u, P_2) = 0$, i.e. $z \in H_0$;
- $\omega_z(u,y) = 0$, a linear condition on y.

Over $S_0 = \{(u,z) : \ u \in \ell_0,\ z \in H_0,\ \beta(u,z) = 0\} \subset \ell_0 \times H_0$ the fiber is a line, and $X = \mathrm{Bl}_{S_0} Z$, with

$$E_1 = -K_Z - (-K_X) = (2a+h) - (a+L+h) = a - L,$$

which is the pull-back of E_{ℓ_0}. Here $S_0 \subset \ell_0 \times H_0 = \mathbb{P}^1 \times \mathbb{P}^2$ is a $(1,1)$-divisor, i.e. $\mathbb{F}_1$.

Description of π_2. The projection π_2 forgets u; it is supported by $L+h$. For fixed (y,z), u must be the kernel of $\omega_z|_{U_3}$,

$$k(y,z) = \omega_z(v_2,w)v_1 - \omega_z(v_1,w)v_2 + \omega_z(v_1,v_2)w,$$

and $\beta(k(y,z),z) = 0$ is an equation of bidegree $(1,2)$, whence $D \subset \mathbb{P}^2_y \times \mathbb{P}^3_z$ (FanoBase IDs 1741, 1126).

The kernel is undefined exactly on $\Sigma = \{\omega_z|_{U_3} = 0\}$, where u only has to satisfy $\beta(u,z) = 0$ on $\mathbb{P}(U_3)$, giving a line. So $X = \mathrm{Bl}_\Sigma D$, with

$$E_2 = -K_D - (-K_X) = (2L+2h) - (a+L+h) = L + h - a.$$

The surface $\Sigma = \{(y,z) : z \in H_0,\ \omega_z(v_1,y) = \omega_z(v_2,y) = 0\} \subset \mathbb{P}^2_y \times H_0$ is cut out by the two $(1,1)$-forms $\omega_z(v_i,y)$, so it is a sextic del Pezzo surface.

Description of π_3. The projection π_3 forgets z; it is supported by $a+L$, with image $W = \mathrm{Bl}_{\ell_0} \mathbb{P}^4$, as in `3-310-67-a`. For $(u, U_3) \in W$ there are three linear conditions on $z \in \mathbb{P}^3$:

$$\beta(u,\cdot), \qquad \omega_\bullet(u,x) \ \ (x \in U_3/u).$$

So the fiber is a point unless they have rank ≤ 2, in which case it is a line. For $u \notin \ell_0$ this is the rank-≤ 2 locus of a 3×4 matrix of linear forms on $\mathbb{P}^4$, of class

$$c_2\big(\mathcal{O}^4 - \mathcal{O}(-1)^3\big) = 6h^2,$$

i.e. a surface $T \simeq \mathrm{Bl}_{10} \mathbb{P}^2$ embedded by quartics. Along ℓ_0 the last two rows are proportional, so $\ell_0 \subset T$, and it is one of the ten exceptional curves of T. Thus

$$X = \mathrm{Bl}_{\tilde{T}} W, \qquad E_3 = -K_W - (-K_X) = (3a+2L) - (a+L+h) = 2a + L - h.$$

Cones. The exceptional divisors above are E_1, E_2 and E_3. The two-dimensional faces $\langle f_1, f_3\rangle$, $\langle f_2, f_3\rangle$ and $\langle f_1, f_2\rangle$ yield respectively the morphisms to $\mathbb{P}^4$, to $\mathbb{P}^2_y$, and to $\mathbb{P}^3_z$; the last is a conic bundle with discriminant H_0 (as $-K_X \cdot h^3 = 2$ and $8 - K_R^2 = 1$ for the preimage R of a line). The classes a, L, h are nef, and f_1, f_2, f_3 are effective and dual to them. Hence $\mathrm{Nef}(X) = \langle a, L, h\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2, f_3\rangle$, with

	f_1	f_2	f_3
a	0	1	0
L	1	0	0
h	0	0	1
E_1	-1	1	0
E_2	1	-1	1
E_3	1	2	-1
$-K$	1	1	1

$\diamond$

References

[1] D. O. Orlov, *Projective bundles, monoidal transformations, and derived categories of coherent sheaves*, Russian Acad. Sci. Izv. Math. **41** (1993), 133–141.

[2] A. A. Beilinson, *Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra*, Funct. Anal. Appl. **12** (1978), 214–216.