Fano 3-173-41-b. $\mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Fl}(1,3,7),\ (V_7/\mathcal{U}_3)^{\oplus 2} \oplus (\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \det \mathcal{U}_3^\vee \otimes \mathcal{U}_1^\vee \oplus \mathcal{O}(1) \boxtimes \mathcal{U}_1^\vee\big)$, with $\mathrm{Fl}(1,3,7) \subset \mathbb{P}^6 \times \mathrm{Gr}(3,7)$.

Invariants. $h^0(-K) = 41,\ (-K)^4 = 173,\ h^{1,1} = 3,\ h^{2,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 9,\ -\chi(T) = 6$.

Description.

(1) $\pi_1 \colon X \to \mathbb{P}^6 \times \mathrm{Gr}(3,7)$ is an extremal divisorial contraction onto the blow-up of $\mathbb{P}^4$ (1-625-126-a) along an elliptic quartic curve (2-305-66-a), the blow-up of it along the strict transform of a quartic del Pezzo surface containing the curve as a hyperplane section.

(2) $\pi_2 \colon X \to \mathbb{P}^1 \times \mathrm{Gr}(3,7)$ is an extremal divisorial contraction onto the blow-up of $\mathbb{P}^4$ along this quartic del Pezzo surface, a divisor of bidegree $(1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^4$ (2-297-65-a): the blow-up of it along the surface $\mathbb{P}^1 \times C$, with C the elliptic quartic curve.

(3) $\pi_3 \colon X \to \mathbb{P}^1 \times \mathbb{P}^6$ is an extremal divisorial contraction onto a singular fourfold: it contracts the strict transform of the hyperplane of $\mathbb{P}^4$ spanned by the elliptic curve onto a curve, with quadric surface fibers.

- t, a, L denote the pullbacks of the hyperplane classes of $\mathbb{P}^1$ and $\mathbb{P}^6$ and of the Plücker class.
- $\mathrm{Nef}(X) = \langle t, a, L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_1 and [Orl]: $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(C), \mathrm{D}^{\mathrm{b}}(C), E_1, \dots, E_{13}\rangle$, with $E_1, \dots, E_{13}$ exceptional objects, coming from $\mathbb{P}^4$ and from the quartic del Pezzo surface.

Explanation.

Write $\mathrm{Fl}(1,3,7) = \{u \in U_3 \subset V_7\}$. The two sections of $V_7/\mathcal{U}_3$ are vectors v_1, v_2, and impose $P_2 = \langle v_1, v_2\rangle \subset U_3$; so $U_3 = P_2 + \langle w\rangle$ with $w \in \mathbb{P}(V_7/P_2) \simeq \mathbb{P}^4$, and we put $\ell_0 = \mathbb{P}(P_2)$. The section of $(\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \det \mathcal{U}_3^\vee \otimes \mathcal{U}_1^\vee$ is a general tensor $\Psi \in S_{221}V_7^\vee \subset \bigwedge^3 V_7^\vee \otimes \bigwedge^2 V_7^\vee$, and imposes that u lies in the kernel of the 2-form $\beta_w = \Psi(v_1 \wedge v_2 \wedge w, \cdot \wedge \cdot)$ restricted to U_3. When $\beta_w|_{U_3} \neq 0$ this kernel is spanned by

$$k(w) = \beta_w(v_2, w)\, v_1 - \beta_w(v_1, w)\, v_2 + \beta_w(v_1, v_2)\, w = Q_2(w)\, v_1 - Q_1(w)\, v_2 + \psi(w)\, w,$$

with ψ linear and Q_1, Q_2 quadratic. The section of $\mathcal{O}(1) \boxtimes \mathcal{U}_1^\vee$ is a general pencil of linear forms λ_t, and imposes $\lambda_t(u) = 0$. Let $C = \{\psi = Q_1 = Q_2 = 0\} \subset \mathbb{P}^4$, a smooth elliptic quartic curve in the hyperplane $\{\psi = 0\}$, and let $S = \{\lambda_0(k(w)) = \lambda_1(k(w)) = 0\}$, the base locus of a pencil of quadrics through C: a quartic del Pezzo surface, with $C = S \cap \{\psi = 0\}$.

Description of π_1. The image of π_1 is the zero locus W of the first three summands in $\mathrm{Fl}(1,3,7)$, the model of 2-305-66-a. Over $w \notin C$ the point $u = k(w)$ is determined, and over $w \in C$ it runs over $\mathbb{P}(U_3) \simeq \mathbb{P}^2$; so $W = \mathrm{Bl}_C \mathbb{P}^4$, as the zero locus of a section of the rank-3 bundle $\bigwedge^2 \mathcal{U}_3^\vee$ twisted by $\mathcal{O}(1)$, vanishing along C. Since $\mathcal{O}(a)$ is globally generated on W, [BFMT, Lemma 3.1] gives $X \simeq \mathrm{Bl}_{S_1} W$, where $S_1 = \{u \in \ker\lambda_0 \cap \ker\lambda_1\}$ is the preimage of the plane $\mathbb{P}(\ker\lambda_0 \cap \ker\lambda_1) \subset \mathbb{P}^6$. Off C this is the condition $w \in S$, and over each point of C it picks one point of $\mathbb{P}(U_3)$; so S_1 is the strict transform of S, isomorphic to S. The fibers f_1 have $t \cdot f_1 = 1$ and $a \cdot f_1 = L \cdot f_1 = 0$.

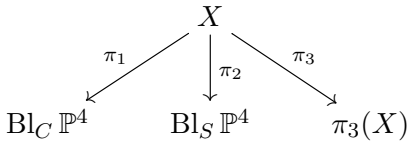
Description of π_2. The image of π_2 is $D = \{(t,w) : \lambda_t(k(w)) = 0\}$, the divisor of bidegree $(1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^4$ defined by the pencil of quadrics through S; so $D = \mathrm{Bl}_S \mathbb{P}^4$, the model of 2-297-65-a. Over (t,w) with $w \notin C$ the point $u = k(w)$ is determined; over $w \in C$, u runs over the line $\mathbb{P}(U_3) \cap \{\lambda_t = 0\}$. So π_2 is birational, with line fibers f_2 over the surface $\mathbb{P}^1 \times C$, and $a \cdot f_2 = 1$, $t \cdot f_2 = L \cdot f_2 = 0$.

Description of π_3. The image of $u = k(w)$ in V_7/P_2 is $\psi(w)\, w$, so w, hence U_3, is recovered from u unless $\psi(w) = 0$. On the hyperplane $\{\psi = 0\}$ we have $u = Q_2(w)v_1 - Q_1(w)v_2 \in \ell_0$, and t is then determined by $\lambda_t(u) = 0$, since $\ell_0 \cap \mathbb{P}(\ker\lambda_0 \cap \ker\lambda_1) = \emptyset$. So π_3 contracts the strict transform of the hyperplane $\{\psi = 0\}$ onto a curve isomorphic to ℓ_0, and the fiber over $u \in \ell_0$ is the quadric surface $\{Q_2 - sQ_1 = 0\} \cap \{\psi = 0\}$ of the pencil, where $u = [s : -1]$ in the basis v_1, v_2. A line f_3 in such a quadric has $L \cdot f_3 = 1$ and $t \cdot f_3 = a \cdot f_3 = 0$.

Cones. The classes t, a, L are nef, and the curves f_1, f_2, f_3 form the dual basis:

	f_1	f_2	f_3
t	1	0	0
a	0	1	0
L	0	0	1

Hence $\mathrm{Nef}(X) = \langle t, a, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2, f_3\rangle$ are dual simplicial cones. The nef class supporting π_i is the sum of the other two, and vanishes exactly on the curves of class proportional to f_i; so each π_i is the contraction of an extremal ray, which is K_X-negative since X is Fano. As π_2 has one-dimensional fibers, [And, Thm. 2.3] gives $X \simeq \mathrm{Bl}_{\mathbb{P}^1 \times C} D$. Finally, X is birational to $\mathbb{P}^4$, hence rational.



◇

References

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