Fano 3-188-44-d. $\mathscr{Z}\big(\mathbb{P}^1 \times \mathrm{Fl}(1,3,5),\ (\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee \oplus (\det \mathcal{U}_3^\vee)^{\oplus 2} \oplus \mathcal{O}(1) \boxtimes \mathcal{U}_1^\vee\big)$.

Invariants. $h^0(-K) = 44,\ (-K)^4 = 188,\ h^{1,1} = 3,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 9,\ -\chi(T) = 5$.

Description. Let $W_5 = \mathrm{Gr}(3, V_5) \cap H_1 \cap H_2$ be the del Pezzo fourfold of degree five (type 1-405-85-a), and let $C \subset W_5$ be the τ-conic of ω-isotropic 3-spaces. The fourfold $F = \mathrm{Bl}_C W_5$ is of type 2-277-61-a.

(1) The projection $\pi_1 \colon X \to F$ is the blow-up of a quintic del Pezzo surface $S_1 \simeq \mathrm{Bl}_4 \mathbb{P}^2$, with exceptional divisor $E_1 = a - h$.
(2) The projection $\pi_2 \colon X \to Y_2$ is the blow-up of $\mathbb{P}^1 \times C$, where $Y_2 \subset \mathbb{P}^1 \times W_5$ is a $(1,1)$-divisor of type 2-240-54-c. Its exceptional divisor is $E_2 = L - a$.
(3) The projection $\pi_3 \colon X \to Z_0$, where $Z_0 \simeq \mathrm{Bl}_{\mathbb{P}^2} \mathbb{P}^4$ is of type 2-512-105-b, is birational, with one-dimensional fibers over the strict transform of a rational quartic scroll and a quadric surface over its improper double point. Its exceptional divisor is $E_3 = 3a - L$.

Here $h = \mathcal{O}_{\mathbb{P}^1}(1)$, $a = \mathcal{U}_1^\vee$, and $L = \det \mathcal{U}_3^\vee$ on X, so that $-K_X = h + a + L$. The three projections define the extremal rays of the simplicial cone $\overline{\mathrm{NE}}(X)$, and $\mathrm{Nef}(X) = \langle h, a, L\rangle$. In particular, X is rational.

Semiorthogonal decompositions. By Orlov's blow-up formula, the successive blow-ups $F = \mathrm{Bl}_C W_5$ and $X = \mathrm{Bl}_{S_1} F$, with $C \simeq \mathbb{P}^1$ of codimension three in W_5 and $S_1 \simeq \mathrm{Bl}_4 \mathbb{P}^2$ of codimension two in F, give

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(W_5), \mathrm{D}^{\mathrm{b}}(C), \mathrm{D}^{\mathrm{b}}(C), \mathrm{D}^{\mathrm{b}}(S_1)\rangle,$$

up to mutations. The two copies of $\mathrm{D}^{\mathrm{b}}(C)$ arise from the codimension-three center. The del Pezzo fourfold W_5 admits a full exceptional collection of length six, while $C \simeq \mathbb{P}^1$ and $S_1 \simeq \mathrm{Bl}_4 \mathbb{P}^2$ admit full exceptional collections of lengths two and seven, respectively, by Beilinson's theorem and Orlov's blow-up formula. Consequently, X admits a full exceptional collection of length $6 + 2 + 2 + 7 = 17 = \chi_{\mathrm{top}}(X)$.

Explanation.

Setup. After embedding $\mathrm{Fl}(1,3,V_5)$ into $\subset \mathbb{P}(V_5) \times \mathrm{Gr}(3,V_5)$, we write $X \subset \mathbb{P}^1 \times \mathbb{P}^4 \times \mathrm{Gr}(3,V_5)$ as

$$X = \{(t, u, U_3) : u \in U_3,\ \omega(u, U_3) = 0,\ U_3 \in H_1 \cap H_2,\ \lambda_t(u) = 0\},$$

where $\omega \in \bigwedge^2 V_5^\vee$ has rank four and kernel $k \in \mathbb{P}^4$, H_1, H_2 are Plücker hyperplanes, and $\lambda_t = t_0\lambda_0 + t_1\lambda_1$ is a pencil of linear forms with base plane $\Pi_K = \mathbb{P}(\ker\lambda_0 \cap \ker\lambda_1)$. As in `2-277-61-a`, F is also the blow-up of $\mathbb{P}^4$ along a projected rational quartic scroll S, having an improper double point at k.

The summands have sections $\bigwedge^2 V_5^\vee$, $\bigwedge^3 V_5^\vee$ (twice) and $(\mathbb{C}^2)^\vee \otimes V_5^\vee$, and they are globally generated. We have

$$c_1(\mathbb{P}^1 \times \mathrm{Fl}(1,3,5)) = 2h + 3a + 4L,$$

and the summands have total first Chern class $(L+a)+2L+(h+a)$, hence $-K_X = h+a+L$.

Description of π_1. The projection π_1 forgets t; it is supported by $a+L$. Its image is

$$F = \mathscr{Z}\big(\mathrm{Fl}(1,3,5), (\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee \oplus (\det \mathcal{U}_3^\vee)^{\oplus 2}\big),$$

which is of type 2-277-61-a; localization on F gives $(-K_F)^4 = 277$ and $\chi_{\mathrm{top}}(F) = 10$. For $(u, U_3) \in F$, the parameter t is determined by $\lambda_t(u) = 0$, unless $u \in \Pi_K$, in which case the fiber is $\mathbb{P}^1$. So $X = \mathrm{Bl}_{S_1} F$, with

$$E_1 = -K_F - (-K_X) = (2a+L) - (h+a+L) = a - h.$$

The map $F \to \mathbb{P}^4$ is an isomorphism off $S \cup \{k\}$, and the plane Π_K meets the quartic surface S in 4 points (and misses k), whence $S_1 = \mathrm{Bl}_4 \mathbb{P}^2$.

Description of π_2. The projection π_2 forgets u; it is supported by $h+L$. For $U_3 \in W_5$ with $\omega|_{U_3} \neq 0$, u must be the kernel line $k(U_3)$ of $\omega|_{U_3}$. The condition $\lambda_t(k(U_3)) = 0$ is, up to a constant, $(\lambda_t \wedge \omega)|_{U_3} = 0$, an equation of bidegree $(1,1)$; this gives Y_2. Projecting $Y_2 \to W_5$ blows up the base locus S_5 of the pencil of hyperplane sections $H_{\lambda_t \wedge \omega}$, a quintic del Pezzo surface. Hence:

- $Y_2 = \mathrm{Bl}_{S_5} W_5$, of type 2-240-54-c;
- $\chi_{\mathrm{top}}(Y_2) = 6 + 7 = 13$ and $h^{2,2}(Y_2) = 2 + 5 = 7$;
- $-K_{Y_2} = h + 2L$, and $(h+2L)^4 = 16 \cdot 5 + 32 \cdot 5 = 240$.

For $U_3 \in C$ (i.e. $\omega|_{U_3} = 0$), every $(\lambda_t \wedge \omega)|_{U_3}$ vanishes, so $C \subset S_5$ and $\mathbb{P}^1 \times C \subset Y_2$. The fiber over (t, U_3) is then $\mathbb{P}(U_3 \cap \ker \lambda_t) \simeq \mathbb{P}^1$. So

$$X = \mathrm{Bl}_{\mathbb{P}^1 \times C} Y_2, \qquad E_2 = (h+2L) - (h+a+L) = L - a.$$

Description of π_3. The projection π_3 forgets U_3; it is supported by $h+a$. Its image is the $(1,1)$-divisor $Z_0 = \{\lambda_t(u) = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^4$, i.e. $\mathrm{Bl}_{\Pi_K} \mathbb{P}^4$, which is of type 2-512-105-b. Over (t, u) the conditions on U_3 are those of the fiber of $F \to \mathbb{P}^4$ over u, so:

- the fibers of π_3 are lines over the strict transform S' of S, and a quadric surface over k;
- $E_3 = -K_{Z_0} - (-K_X) = (h+4a) - (h+a+L) = 3a - L$, which is the pull-back of the exceptional divisor of $F \to \mathbb{P}^4$.

Since S meets Π_K in 4 points, the normalization of S' is $\mathrm{Bl}_4(\mathbb{P}^1 \times \mathbb{P}^1)$.

Cones. The two-dimensional faces of $\overline{\mathrm{NE}}(X)$ correspond to the morphisms $X \to \mathbb{P}^4$ (type 1-625-126-a), $X \to W_5$ (type 1-405-85-a), and $X \to \mathbb{P}^1$. The last map has fibers given by the preimages under $F \to \mathbb{P}^4$ of the hyperplanes $\{\lambda_t = 0\}$, namely $\mathbb{P}^3$ blown up along a rational quartic curve. Since X is birational to W_5, it is rational.

The classes h, a, L are nef, and f_1, f_2, f_3 are effective and dual to them. Hence $\mathrm{Nef}(X) = \langle h, a, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2, f_3\rangle$, with

	f_1	f_2	f_3
h	1	0	0
a	0	1	0
L	0	0	1
$E_1 = a - h$	-1	1	0
$E_2 = L - a$	0	-1	1
$E_3 = 3a - L$	0	3	-1
$-K$	1	1	1

$\diamond$