Fano 3-193-45-d. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(2,5),\ \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \bigwedge^2 \mathcal{Q} \ \oplus\ \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee\big)$.

Invariants. $h^0(-K) = 45,\ (-K)^4 = 193,\ h^{1,1} = 3,\ h^{2,1} = 1,\ h^{3,1} = 0,\ h^{2,2} = 5,\ -\chi(T) = 2$.

Description.

(1) $\pi_1 \colon X \to \mathbb{P}^3 \times \mathrm{Gr}(2,5)$ is an extremal small contraction: it contracts a plane with normal bundle $\mathcal{O}(-1)^{\oplus 2}$ to a point, and is an isomorphism elsewhere.
(2) $\pi_2 \colon X \to \mathbb{P}^2 \times \mathrm{Gr}(2,5)$ is an extremal divisorial contraction onto $Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2})$ (2-513-105-b), the blow-up of Y along a surface isomorphic to the symmetric square $C^{(2)}$ of an elliptic quartic curve $C \subset \mathbb{P}^3$.
(3) $\pi_3 \colon X \to \mathbb{P}^2 \times \mathbb{P}^3$ is an extremal divisorial contraction onto a smooth divisor D of bidegree $(1,1)$ (2-432-90-d), the blow-up of D along a $\mathbb{P}^1$-bundle over C.

- z, x, L denote the pullbacks of the hyperplane classes of $\mathbb{P}^2$, $\mathbb{P}^3$ and of the Plücker class; $\mathrm{Nef}(X) = \langle z, x, L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_3 and [Orl], since D is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$:

$$\mathrm{D^b}(X) = \langle \mathrm{D^b}(C), \mathrm{D^b}(C), E_1, \dots, E_9\rangle,$$

  with $E_1, \dots, E_9$ exceptional objects.

Explanation.

Write $\mathbb{P}^2 = \mathbb{P}(V_3)$, $\mathbb{P}^3 = \mathbb{P}(V_4)$ and $\mathrm{Gr}(2,5) = \mathrm{Gr}(2, V_5)$. The section of $\mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \bigwedge^2 \mathcal{Q}$ is a general linear map $b \colon V_4 \to \bigwedge^2 V_5$, $x \mapsto b_x$, and vanishes at (x, U) if and only if $b_x \in U \wedge V_5$, that is, $U^\perp \subset V_5^\vee$ is isotropic for the skew form b_x. The section of $\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee$ is a surjection $A \colon V_5 \to V_3$, and vanishes at (z, U) if and only if $A(U) \subset z$. Let $K = \ker A$. When b_x has rank 4, its kernel is spanned by $b_x \wedge b_x \in \bigwedge^4 V_5 \simeq V_5^\vee$. Let $C \subset \mathbb{P}^3$ be the locus where $b_x \wedge b_x$ vanishes on K: it is a complete intersection of two quadrics, a smooth elliptic quartic curve for general b, and it contains the five points where b_x has rank 2.
Description of π_1. The summand $\mathcal{O}(1) \boxtimes \bigwedge^2 \mathcal{Q}$ does not involve $\mathbb{P}^2$, and $\pi_{1*}\mathcal{Q}_{\mathbb{P}^2} = V_3 \otimes \mathcal{O}$ with vanishing higher direct images. Hence the fiber of π_1 over (x, U) is the zero locus in $\mathbb{P}^2$ of the composite $U \otimes \mathcal{O} \xrightarrow{A} V_3 \otimes \mathcal{O} \to \mathcal{Q}_{\mathbb{P}^2}$, namely $\{z : A(U) \subset z\}$: it is empty if $A|_U$ has rank 2, the reduced point $[A(U)]$ if it has rank 1, and $\mathbb{P}^2$ if $U = K$. Therefore $\pi_1(X) = \mathscr{Z}(\mathbb{P}^3 \times \Sigma, \mathcal{O}(1) \boxtimes \bigwedge^2 \mathcal{Q})$, where $\Sigma = D_1(A|_{\mathcal{U}}) = \{U \cap K \neq 0\}$ is the Schubert variety of class $c_2(\mathcal{Q}) = \sigma_2$, and π_1 is an isomorphism outside $\mathbb{P}^3 \times \{K\}$. There the remaining condition is the vanishing of $V_4 \xrightarrow{b} \bigwedge^2 V_5 \to \bigwedge^2 (V_5/K)$, a general map of rank 3, whose kernel is a point x_0. So π_1 contracts exactly the plane $P = \mathbb{P}^2 \times \{(x_0, K)\}$.

Along P the differential of the section is block triangular: on $T_{x_0}\mathbb{P}^3$ it is the isomorphism $V_4/x_0 \simeq \bigwedge^2(V_5/K)$ induced by b, and on $\operatorname{Hom}(K, V_5/K) \simeq K^\vee \otimes V_3$ it is the surjection onto $K^\vee \otimes \mathcal{Q}_{\mathbb{P}^2}$. Hence $N_{P/X} \simeq K^\vee \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \simeq \mathcal{O}(-1)^{\oplus 2}$. A line $r_1 \subset P$ has $z \cdot r_1 = 1$ and $x \cdot r_1 = L \cdot r_1 = 0$; the nef class $x + L$ vanishes exactly on the curves of P, so r_1 spans an extremal ray, with $-K_X \cdot r_1 = 1$. The image $\pi_1(X)$ is cut out in the Cohen–Macaulay variety $\mathbb{P}^3 \times \Sigma$ by three equations in the expected codimension, and is smooth outside one point, hence normal: π_1 is the contraction of this ray.

Description of π_2. The second section cuts out $Y = \{(z, U) : U \subset W_3\}$, where $W_3 = A^{-1}(z) \simeq K \otimes \mathcal{O} \oplus \mathcal{O}(-1)$ on $\mathbb{P}^2$. Thus $Y = \mathbb{P}_{\mathbb{P}^2}(W_3^\vee) \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-1)^{\oplus 2})$ is the bundle of hyperplanes of W_3, with relative hyperplane class $L - z$; this is the model of 2-513-105-b.

The summand $\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee$ cuts out Y, and $\pi_{2*}\mathcal{O}_{\mathbb{P}^3}(1) = V_4^\vee \otimes \mathcal{O}$. Hence the other summand gives the morphism

$$\varphi \colon V_4 \otimes \mathcal{O}_Y \longrightarrow \bigwedge^2 Q \cong \bigwedge^2(V_5/\mathcal{U}), \qquad x \longmapsto b_x \bmod \mathcal{U} \wedge V_5,$$

and the fiber of π_2 over $y \in Y$ is $\mathbb{P}(\ker \varphi_y)$. Since $\mathcal{H}om(V_4 \otimes \mathcal{O}, \bigwedge^2 \mathcal{Q})$ is generated by $V_4^\vee \otimes \bigwedge^2 V_5$ and b is general, the loci $D_k(\varphi)$ have the expected codimension $(4-k)(3-k)$ and $D_k(\varphi) \setminus D_{k-1}(\varphi)$ is smooth. So $D_1(\varphi) = \emptyset$, and φ drops rank exactly along the smooth surface $S_2 = D_2(\varphi)$, of class $(c_1^2 - c_2)(\bigwedge^2 \mathcal{Q})|_Y = (2\sigma_2 + 3\sigma_{1,1})|_Y$ [Ful, Thm. 14.4]. Near a point of S_2, two rows and columns of φ split off and leave a map $\mathcal{O}^{\oplus 2} \to \mathcal{O}$ given by local equations f_0, f_1 of S_2; thus X is locally $\{f_0 t_1 - f_1 t_0 = 0\}$ in a $\mathbb{P}^1$-bundle over Y, and $X \simeq \operatorname{Bl}_{S_2} Y$ by [BFMT, Lemma 3.1]. A line r_2 in a fiber over S_2 has $x \cdot r_2 = 1$, $z \cdot r_2 = L \cdot r_2 = 0$ and $-K_X \cdot r_2 = 1$; the nef class $z + L$ vanishes exactly on these lines, so r_2 spans an extremal ray and π_2 is its contraction. The exceptional divisor is $E_2 = -K_Y + K_X = 2L - x$.

By [BFMT, Proposition 3.6], $\mathcal{O}_{S_2}$ is resolved by

$$0 \longrightarrow \mathcal{Q}(-3L)|_Y \longrightarrow V_4^\vee \otimes \mathcal{O}_Y(-2L) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{S_2} \longrightarrow 0.$$

On the $\mathbb{P}^2$-bundle $Y \to \mathbb{P}^2$ the middle term is acyclic, and the only cohomology of $\mathcal{Q}(-3L)|_Y$ is $H^3 = H^1(\mathbb{P}^2, \Omega^1) = \mathbb{C}$. Hence S_2 is connected, so irreducible, with $q(S_2) = 1$ and $p_g(S_2) = 0$. For $y = (z, U) \in S_2$, every b_x with x on the line $\ell_y = \mathbb{P}(\ker \varphi_y)$ has the 3-dimensional isotropic subspace $U^\perp$, which therefore contains the kernel $b_x \wedge b_x$. As $U^\perp \cap K^\perp = W_3^\perp$ is a line in $\mathbb{P}(U^\perp)$, the line ℓ_y meets C in a scheme of length two: ℓ_y is a secant of C. This defines a morphism $S_2 \to C^{(2)}$, which is dominant because E_2 dominates $\mathbb{P}^3$. Over a general $x \in \mathbb{P}^3$ the fiber of $X \to \mathbb{P}^3$ is a line of $\operatorname{Gr}(2,5)$, namely the planes U that are Lagrangian for b_x and contain the line $K \cap \operatorname{im} b_x$; hence $E_2 \cdot x^3 = (2L - x) \cdot x^3 = 2$, the number of secants of C through a general point, and $S_2 \to C^{(2)}$ is birational. Since $h^{1,1}(S_2) = h^{2,2}(X) - h^{2,2}(Y) = 2 = h^{1,1}(C^{(2)})$, it is an isomorphism.

Description of π_3. Over $\mathbb{P}^2$, Y is the bundle of hyperplanes U of W_3, so X lies in the $\mathbb{P}^2$-bundle $\mathbb{P}(W_3^\vee)$ over $\mathbb{P}^2 \times \mathbb{P}^3$. There $\bigwedge^2(V_5/\mathcal{U})$ has the quotient $\bigwedge^2(V_5/W_3) \simeq \mathcal{O}(z)$, and the corresponding component of the equation is the composite

$$V_4 \xrightarrow{b} \bigwedge^2 V_5 \longrightarrow \bigwedge^2(V_5/K) \simeq \bigwedge^2 V_3 \simeq V_3^\vee,$$

a 3×4 matrix of rank 3. Hence $\pi_3(X)$ lies in the smooth divisor D of bidegree $(1,1)$ defined by this bilinear form; its projection $D \to \mathbb{P}^2$ is a $\mathbb{P}^2$-bundle. On D we have $b_x \in W_3 \wedge V_5$, and writing $U^\perp = W_3^\perp + \langle m\rangle$, the remaining equation $b_x(W_3^\perp, m) = 0$ is the morphism

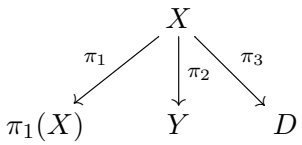
$$\psi\colon W_3^\vee \longrightarrow (W_3^\perp)^\vee \otimes \mathcal{O}(x) = \mathcal{Q}_{\mathbb{P}^2}(x),$$

whose kernel gives the fiber of π_3. It drops rank exactly where $\ker b_x \cap W_3^\perp \neq 0$: for b_x of rank 4 this means that $b_x \wedge b_x \in W_3^\perp$, that is, $x \in C$ and z lies on a line; at the five points where b_x has rank 2 the 3-dimensional kernel meets $K^\perp$ in a line, and again z lies on a line. It never vanishes, since $b_x \in \bigwedge^2 W_3$ would force a decomposable b_x whose plane meets K. So $S_3 = D_1(\psi)$ is a $\mathbb{P}^1$-bundle over C, of the expected codimension 2, with class $(c_1^2 - c_2)(\mathcal{Q}(x) - W_3^\vee)|_D = (3x^2 + zx - z^2)|_D$. Locally ψ reduces to $\mathcal{O}^{\oplus 2} \to \mathcal{O}$ given by equations f_0, f_1 of S_3, and X is $\{f_0 t_1 - f_1 t_0 = 0\}$, so $X \simeq \mathrm{Bl}_{S_3} D$ by [BFMT, Lemma 3.1]; S_3 is smooth because X is. A line r_3 in a fiber over S_3 has $L\cdot r_3 = 1$ and $z\cdot r_3 = x\cdot r_3 = 0$, and the nef class $z+x$ vanishes exactly on these lines: π_3 is the contraction of the extremal ray of r_3. In particular X is birational to D, hence rational.

Cones. The classes z, x, L are nef, and the curves r_1, r_2, r_3 form the dual basis:

	r_1	r_2	r_3
z	1	0	0
x	0	1	0
L	0	0	1

Hence $\mathrm{Nef}(X) = \langle z, x, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle r_1, r_2, r_3\rangle$ are dual simplicial cones.



◇

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