Fano 3-199-46-b. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathbb{P}^4 \times \mathrm{Gr}(2,4),\ (\mathcal{O}_{\mathbb{P}^4}(1) \boxtimes \mathcal{U}^\vee)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{Q}\big)$.

Invariants. $h^0(-K) = 46,\ (-K)^4 = 199,\ h^{1,1} = 3,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 5,\ -\chi(T) = 0$.

Description.

(1) $\pi_1 \colon X \to \mathbb{P}^4 \times \mathrm{Gr}(2,4)$ is an extremal divisorial contraction onto the blow-up of $\mathbb{P}^4$ along a rational normal quartic curve (2-273-60-a), the blow-up of it along a surface $\mathbb{F}_1$.
(2) $\pi_2 \colon X \to \mathbb{P}^2 \times \mathrm{Gr}(2,4)$ is an extremal divisorial contraction onto the blow-up of the quadric $Q^4 = \mathrm{Gr}(2,4)$ along a plane (2-432-90-a), the blow-up of it along the strict transform of a Veronese surface meeting the plane in one point, a surface $\mathbb{F}_1$.
(3) $\pi_3 \colon X \to \mathbb{P}^2 \times \mathbb{P}^4$ is an extremal divisorial contraction onto the blow-up of $\mathbb{P}^4$ along a cubic scroll (2-351-75-a), the blow-up of it along a $\mathbb{P}^1$-bundle over a rational normal quartic curve.

- h, H, L denote the pullbacks of the hyperplane classes of $\mathbb{P}^2$ and $\mathbb{P}^4$ and of the Plücker class.
- $\mathrm{Nef}(X) = \langle h, H, L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_3 and [Orl], $\mathrm{D}^{\mathrm{b}}(X)$ has a full exceptional collection of length $5+4+4 = 13$.

Explanation.

Write $\mathbb{P}^2 = \mathbb{P}(A_3)$, $\mathbb{P}^4 = \mathbb{P}(V_5)$ and $\mathrm{Gr}(2,4) = \mathrm{Gr}(2, W_4) = Q^4$. The two sections of $\mathcal{O}(1) \boxtimes \mathcal{U}^\vee$ are general bilinear forms $B_1, B_2 \colon V_5 \times W_4 \to \mathbb{C}$, and impose $B_1(x,U) = B_2(x,U) = 0$; let $K(x) = \{w : B_1(x,w) = B_2(x,w) = 0\}$, of dimension at least 2. The section of $\mathcal{O}(1) \boxtimes \mathcal{Q}$ is a general injection $\psi \colon A_3 \to W_4$, with image a hyperplane W_3, and imposes $\psi(z) \in U$. Thus

$$X = \{(z, x, U) : \psi(z) \in U \subset K(x)\}.$$

The space $K(x)$ has dimension 3 exactly where the 2×4 matrix of linear forms $(B_i(x,\cdot))$ has rank one, a curve of degree 4 [Ful, Thm. 14.4], the rational normal quartic C; it never has rank zero.

Description of π_1. The image of π_1 is $Y_1 = \{(x,U) : U \subset K(x)\}$, that is

$$Y_1 = \mathscr{Z}\big(\mathbb{P}^4 \times \mathrm{Gr}(2,4), (\mathcal{O}(1) \boxtimes \mathcal{U}^\vee)^{\oplus 2}\big),$$

the model of the family 2-273-60-a. Its projection to $\mathbb{P}^4$ is an isomorphism off C, with fibers $\mathrm{Gr}(2, K(x)) \simeq \mathbb{P}^2$ over C, so $Y_1 = \mathrm{Bl}_C \mathbb{P}^4$. Its projection to Q^4 has as fiber over U the kernel of $V_5 \to U^\vee \oplus U^\vee$: a point, except for a line over the surface S where this map has rank 3, which is the Veronese surface of the secant lines of C. The fiber of π_1 over (x, U) is $\{z : \psi(z) \in U \cap W_3\}$: a point if $U \not\subset W_3$, and a line if $U \subset W_3$. So π_1 is birational, with line fibers over the preimage S_1 of the plane $\mathrm{Gr}(2, W_3) \subset Q^4$. This plane meets S transversally in one point, so S_1 is the plane blown up at one point, $\mathbb{F}_1$. The fibers f_1 have $h \cdot f_1 = 1$ and $H \cdot f_1 = L \cdot f_1 = 0$.

Description of π_2. The image of π_2 is

$$G = \{(z, U) : \psi(z) \in U\} = \mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2, 4), \mathcal{O}(1) \boxtimes \mathcal{Q}\big),$$

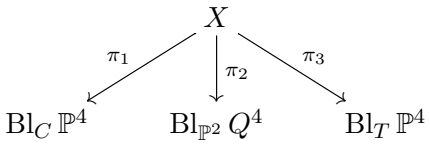
the model of 2-432-90-a under $\mathrm{Gr}(2, W_4) \simeq \mathrm{Gr}(2, W_4^\vee)$. Its projection to $\mathbb{P}^2$ is the $\mathbb{P}^2$-bundle of the planes containing $\psi(z)$, and its projection to Q^4 is an isomorphism off the plane $\mathrm{Gr}(2, W_3)$, over which the fibers are lines; so $G = \mathrm{Bl}_{\mathrm{Gr}(2,W_3)} Q^4$. The fiber of π_2 over (z, U) is the kernel of $V_5 \to U^\vee \oplus U^\vee$, a point unless $U \in S$, and a line over the preimage S_2 of S, which is the strict transform of S, blown up at its point on $\mathrm{Gr}(2, W_3)$: again $\mathbb{F}_1$. The fibers f_2 have $H \cdot f_2 = 1$ and $h \cdot f_2 = L \cdot f_2 = 0$.

Description of π_3. For $x \notin C$, $U = K(x)$ is determined, and $\psi(z) \in K(x)$ gives the two equations $B_1(x, \psi(z)) = B_2(x, \psi(z)) = 0$ of bidegree $(1, 1)$. So the image of π_3 is $Z = \mathscr{Z}(\mathbb{P}^2 \times \mathbb{P}^4, \mathcal{O}(1, 1)^{\oplus 2})$, the model of 2-351-75-a. Its projection to $\mathbb{P}^4$ has as fiber over x the kernel of the 2×3 matrix $(B_i(x, \psi(\cdot)))$: a point, except for a line over the cubic scroll T where it has rank one, so $Z = \mathrm{Bl}_T \mathbb{P}^4$. For $x \in C$, $K(x) \cap W_3$ has dimension 2, so z runs over a line, and then U runs over the planes of $K(x)$ containing $\psi(z)$, again a line. So π_3 is birational, with line fibers over the surface $S_3 = \{(z, x) : x \in C,\ \psi(z) \in K(x)\}$, a $\mathbb{P}^1$-bundle over C. The fibers f_3 have $L \cdot f_3 = 1$ and $h \cdot f_3 = H \cdot f_3 = 0$.

Contractions and cones. The classes h, H, L are nef, and the curves f_1, f_2, f_3 form the dual basis:

	f_1	f_2	f_3
h	1	0	0
H	0	1	0
L	0	0	1

Hence $\mathrm{Nef}(X) = \langle h, H, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2, f_3\rangle$ are dual simplicial cones. Each π_i is supported by the sum of the two nef classes other than the i-th one, contracts exactly the curves of class proportional to f_i, and has fibers of dimension at most one onto a smooth target. Since X is Fano, π_i is the contraction of a K_X-negative extremal ray, and [And, Thm. 2.3] shows that it is the blow-up of the smooth surface described above. In particular X is rational.



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