Fano 3-241-54-e. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Fl}(1,4,6),\ (\mathcal{U}_1^\vee)^{\oplus 2} \oplus \bigwedge^2(\mathcal{U}_4/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee \oplus \mathcal{Q}_{\mathbb{P}^2} \boxtimes V_6/\mathcal{U}_4\big)$, with $\mathrm{Fl}(1,4,6) \subset \mathbb{P}^5 \times \mathrm{Gr}(4,6)$.

Invariants. $h^0(-K) = 54,\ (-K)^4 = 241,\ h^{1,1} = 3,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 5,\ -\chi(T) = -2$.

Description.

(1) $\pi_1 \colon X \to \mathbb{P}^5 \times \mathrm{Gr}(4,6)$ is an extremal small contraction: it contracts a plane with normal bundle $\mathcal{O}(-1)^{\oplus 2}$ to a point, and is an isomorphism elsewhere.
(2) $\pi_2 \colon X \to \mathbb{P}^2 \times \mathrm{Gr}(4,6)$ is an extremal divisorial contraction onto a special member Y of the family 2-352-75-b, the blow-up of Y along a surface isomorphic to $\mathbb{P}^2$.
(3) $\pi_3 \colon X \to \mathbb{P}^2 \times \mathbb{P}^5$ is an extremal divisorial contraction onto a smooth divisor Z of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ (2-432-90-d), the blow-up of Z along a surface $\mathbb{P}^1 \times \mathbb{P}^1$, a $\mathbb{P}^1$-bundle over a twisted cubic curve of $\mathbb{P}^3$.

- z, x, L denote the pullbacks of the hyperplane classes of $\mathbb{P}^2$, $\mathbb{P}^5$ and of the Plücker class.
- $\mathrm{Nef}(X) = \langle z, x, L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_3 and [Orl], since Z is a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$, $\mathrm{D}^{\mathrm{b}}(X)$ has a full exceptional collection of length $9 + 4 = 13$.

Explanation.

Write $\mathbb{P}^2 = \mathbb{P}(V_3)$, $\mathbb{P}^5 = \mathbb{P}(V_6)$ and $\mathrm{Gr}(4,6) = \mathrm{Gr}(4,V_6)$, so that X consists of triples (z,u,U_4) with $u \subset U_4$. The two sections of $\mathcal{U}_1^\vee$ are linear forms with common kernel K_4, and impose $u \in K_4$. The section of $\bigwedge^2(\mathcal{U}_4/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee$ is a general 3-form $\sigma = \mathrm{vol}_A + \mathrm{vol}_B$, where $V_6 = A \oplus B$ with $\dim A = \dim B = 3$, and imposes $\sigma(u, U_4, U_4) = 0$. The section of $\mathcal{Q}_{\mathbb{P}^2} \boxtimes V_6/\mathcal{U}_4$ is a general injection $\varphi \colon V_3^\vee \to V_6$, with image Λ, and imposes $\Lambda_z := \varphi(z^\perp) \subset U_4$. We write $u = a_u + b_u$ with $a_u \in A$, $b_u \in B$; for general data, the projections $\Lambda \to A$ and $\Lambda \to B$ are isomorphisms.

Since $\bigwedge^3 U_4^\vee \simeq U_4 \otimes \det U_4^\vee$, the restriction $\sigma|_{U_4}$ defines a section s_σ of $\mathcal{U}_4 \otimes \det \mathcal{U}_4^\vee$: when $\sigma|_{U_4} \neq 0$, the condition $\sigma(u, U_4, U_4) = 0$ means that u is the line spanned by $s_\sigma(U_4)$. Moreover $\sigma|_{U_4} = 0$ if and only if $U_4 = A' \oplus B'$ with $A' \subset A$ and $B' \subset B$ of dimension 2. Finally, by [BFMT, Proposition 3.3],

$$G = \{(z, U_4) : \Lambda_z \subset U_4\} = \mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(4,6), \mathcal{Q}_{\mathbb{P}^2} \boxtimes V_6/\mathcal{U}_4\big)$$

is a $\mathrm{Gr}(2,4)$-bundle over $\mathbb{P}^2$.

Description of π_1. The fiber of π_1 over (u, U_4) is the zero locus in $\mathbb{P}^2$ of the composite $z^\perp \otimes \mathcal{O} \to V_3^\vee \otimes \mathcal{O} \xrightarrow{\varphi} V_6/U_4 \otimes \mathcal{O}$, namely $\{z : \Lambda_z \subset U_4\}$: it is empty if $\Lambda \to V_6/U_4$ has rank 2, a reduced point if it has rank 1, and $\mathbb{P}^2$ if $\Lambda \subset U_4$. So π_1 is an isomorphism over the complement of $\{\Lambda \subset U_4\}$, and its image lies over $\Sigma = D_1(\Lambda \otimes \mathcal{O} \to V_6/\mathcal{U}_4) = \{\dim(U_4 \cap \Lambda) \geq 2\}$, of class $\sigma_{1,1}$ [Ful, Thm. 14.4]. On the plane $\{U_4 \supset \Lambda\} \simeq \mathbb{P}(V_6/\Lambda)$ we have $\mathcal{U}_4 \otimes \det \mathcal{U}_4^\vee \subset V_6 \otimes \mathcal{O}(1)$, so $U_4 \mapsto s_\sigma(U_4)$ is a linear map; it is injective, since $\sigma|_{U_4} \neq 0$ when $U_4 \supset \Lambda$. Its image is a plane of $\mathbb{P}^5$, which meets $\mathbb{P}(K_4)$ in a single point $u_0 = s_\sigma(U_4^0)$. Hence π_1 contracts exactly the plane $\Pi = \mathbb{P}^2 \times \{(u_0, U_4^0)\}$. The image $\pi_1(X)$ is cut out in the Cohen–Macaulay variety $\mathbb{P}(\mathcal{U}_4)|_\Sigma$ by five equations in the expected codimension, and is smooth outside one point, hence normal. A line $f_1 \subset \Pi$ has $z \cdot f_1 = 1$ and $x \cdot f_1 = L \cdot f_1 = 0$, and the nef class $x + L$ vanishes exactly on the curves of Π; so π_1 is the contraction of the extremal ray spanned by f_1, which is K_X-negative since X is Fano. By [Kaw], $N_{\Pi/X} \simeq \mathcal{O}(-1)^{\oplus 2}$.

Description of π_2. The image of π_2 lies in G, and the fiber over $(z, U_4) \in G$ is $\{u \in \mathbb{P}(U_4 \cap K_4) : \sigma(u, U_4, U_4) = 0\}$. Outside the zero locus S_2 of s_σ, u must be $s_\sigma(U_4)$, which lies in K_4 exactly when the two Plücker linear forms $\sigma \wedge \lambda_1, \sigma \wedge \lambda_2 \in \bigwedge^4 V_6^\vee$ vanish on U_4, where λ_1, λ_2 are the two linear forms. Hence

$$\pi_2(X) = Y = G \cap H_{\sigma\wedge\lambda_1} \cap H_{\sigma\wedge\lambda_2} = \mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(4,6), \mathcal{Q}_{\mathbb{P}^2} \boxtimes V_6/\mathcal{U}_4 \oplus \mathcal{O}(L)^{\oplus 2}\big),$$

the model of 2-352-75-b, for a special section: the two hyperplane sections contain S_2. On S_2 we have $U_4 = A' \oplus B'$, and $\Lambda_z \subset U_4$ forces A' and B' to be the projections of Λ_z to A and B; hence $S_2 \simeq \mathbb{P}^2_z$, and the fiber of π_2 over it is the line $\mathbb{P}(U_4 \cap K_4)$. The exceptional divisor $\pi_2^{-1}(S_2)$ is therefore a $\mathbb{P}^1$-bundle over S_2. Its fibers f_2 have $x \cdot f_2 = 1$ and $z \cdot f_2 = L \cdot f_2 = 0$, and the nef class $z + L$ vanishes exactly on them; so π_2 is the contraction of the extremal ray spanned by f_2, which is K_X-negative and has one-dimensional fibers. By [And, Thm. 2.3], Y is smooth and $X \simeq \mathrm{Bl}_{S_2} Y$.

Description of π_3. For (z, u) with $u \notin \Lambda_z$, put $W = \Lambda_z + \langle u \rangle$, so that $U_4 = W + \langle m \rangle$ with $m \in \mathbb{P}(V_6/W) \simeq \mathbb{P}^2$. The condition $\sigma(u, U_4, U_4) = 0$ splits into $\sigma(u, \Lambda_z, \Lambda_z) = 0$, which does not involve m, and the morphism

$$V_6/W \longrightarrow \Lambda_z^\vee, \qquad m \longmapsto \sigma(u, \cdot, m)|_{\Lambda_z},$$

of ranks $3 \to 2$, whose kernel gives the fiber. The first condition is the bilinear form defined by

$$K_4 \subset V_6 \longrightarrow (\textstyle\bigwedge^2 \Lambda)^\vee \simeq V_3, \qquad u \longmapsto \sigma(u, \cdot, \cdot)|_\Lambda,$$

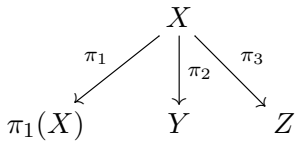
a 3×4 matrix of rank 3. Hence $\pi_3(X)$ is the smooth divisor Z of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}(K_4)$ defined by this form, whose projection to $\mathbb{P}^2$ is a $\mathbb{P}^2$-bundle; it is the model of 2-432-90-d. The morphism above never vanishes, and drops rank exactly where Λ_z meets the kernel $\langle a_u, b_u \rangle$ of the 2-form $\iota_u \sigma$. If $u \in \Lambda_z$, which only happens for the point u of $\mathbb{P}(\Lambda) \cap \mathbb{P}(K_4)$ and z on a line, then U_4/u runs over the 3-dimensional isotropic subspaces of the rank-4 form $\iota_u \sigma$ containing Λ_z/u, again a line.

For each $(\alpha : \beta) \in \mathbb{P}^1$ there is exactly one $u \in \mathbb{P}(K_4)$ with $\alpha a_u + \beta b_u \in \Lambda$. These points trace a twisted cubic $\Gamma \subset \mathbb{P}(K_4)$, through $\mathbb{P}(A) \cap \mathbb{P}(K_4)$ and $\mathbb{P}(B) \cap \mathbb{P}(K_4)$, and the z with $\alpha a_u + \beta b_u \in \Lambda_z$ form a line depending quadratically on $(\alpha : \beta)$. So the locus S_3 where the fiber is a line is a divisor of bidegree $(2, 1)$ in $\Gamma \times \mathbb{P}^2$, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Its fibers f_3 are lines in $\mathrm{Gr}(4, 6)$, with $L \cdot f_3 = 1$ and $z \cdot f_3 = x \cdot f_3 = 0$, and the nef class $z + x$ vanishes exactly on them. As before, [And, Thm. 2.3] gives $X \simeq \mathrm{Bl}_{S_3} Z$. In particular X is birational to Z, hence rational.

Cones. The classes z, x, L are nef, and the curves f_1, f_2, f_3 form the dual basis:

	f_1	f_2	f_3
z	1	0	0
x	0	1	0
L	0	0	1

Hence $\mathrm{Nef}(X) = \langle z, x, L \rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2, f_3 \rangle$ are dual simplicial cones.



◇

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