Fano 3-262-58-a. $X = \mathscr{Z}(\mathbb{P}^2 \times \mathrm{Fl}(1,3,4),\ \mathcal{U}_1^\vee \oplus \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes (\mathcal{U}_1^\vee \otimes (\mathcal{U}_3/\mathcal{U}_1)^\vee))$.

Invariants. $h^0(-K_X) = 58, \quad (-K_X)^4 = 262, \quad h^{1,1} = 3, \quad h^{2,2} = 5, \quad -\chi(T_X) = -3.$

Description. Regard the ambient flag as an incidence in $\mathbb{P}^3 \times \mathrm{Gr}(3,4)$, and let π_i omit the i-th factor of $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathrm{Gr}(3,4)$. Let h, a, L be the corresponding hyperplane classes. Write F, G for two isomorphic $(1,1)$ divisors in $\mathbb{P}^2 \times \mathbb{P}^3$, of type 2-432-90-d.

(1) $\pi_1 \colon X \simeq \mathrm{Bl}_S F \to F$ is a K-negative extremal divisorial contraction, with centre a smooth quadric surface $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$.
(2) $\pi_2 \colon X \simeq \mathrm{Bl}_{S'} G \to G$ is a K-negative extremal divisorial contraction, with centre another smooth quadric surface $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$.
(3) $\pi_3 \colon X \simeq \mathrm{Bl}_C(\mathbb{P}^2 \times \mathbb{P}^2) \to \mathbb{P}^2 \times \mathbb{P}^2$ is a K-negative extremal divisorial contraction, with centre a smooth curve $C \simeq \mathbb{P}^1$ of bidegree $(2,2)$.

- $\mathrm{Nef}(X) = \langle h, a, L \rangle$; its facets induce the extremal contractions π_1, π_2, π_3.
- X is rational.

Semiorthogonal decompositions. Orlov's blow-up formula [Orl] gives, up to the standard embeddings and twists,

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(C), \mathrm{D}^{\mathrm{b}}(C), \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2 \times \mathbb{P}^2) \rangle.$$

Since $C \simeq \mathbb{P}^1$, X has a full exceptional collection of length 13.

Explanation. Write $\mathbb{P}^2 = \mathbb{P}(V_3)$ and $\mathbb{P}^3 = \mathbb{P}(V_4)$. The first section fixes a hyperplane $\Lambda \subset V_4$, so X parametrizes triples (x, u, U_3) with $x \in \mathbb{P}(V_3)$, $u \in \mathbb{P}(\Lambda)$, $u \subset U_3$, and $\omega_x(u, U_3) = 0$. Indeed, $H^0(\mathrm{Fl}(1,3,V_4), \mathcal{U}_1^\vee \otimes (\mathcal{U}_3/\mathcal{U}_1)^\vee) \simeq \bigwedge^2 V_4^\vee$, so the remaining section is a general net $x \mapsto \omega_x$ of alternating forms. Its restriction to Λ yields an isomorphism $V_3 \xrightarrow{\sim} \bigwedge^2 \Lambda^\vee$, whose radicals define a projective linear isomorphism $p \colon \mathbb{P}(V_3) \xrightarrow{\sim} \mathbb{P}(\Lambda)$. The net meets the Klein quadric along a smooth conic $\Delta \subset \mathbb{P}(V_3)$, and the lines $\mathbb{P}(\ker \omega_x)$, $x \in \Delta$, form one ruling of a smooth quadric $Q_0 \subset \mathbb{P}(V_4)$. Let $\Gamma = p(\Delta) = Q_0 \cap \mathbb{P}(\Lambda)$. For $v \in Q_0$, denote by $K(v) = \mathbb{P}(\ker \omega_{x(v)})$ and $N(v)$ the two ruling lines through v, and set $u(v) = N(v) \cap \mathbb{P}(\Lambda)$. The images of the first two projections are the incidence fourfolds

$$F = \{(u, U_3) : u \in \mathbb{P}(\Lambda),\ u \subset U_3\}, \qquad G = \{(x, U_3) : p(x) \subset U_3\},$$

both $(1,1)$ divisors in $\mathbb{P}^2 \times \mathbb{P}^3$. Adjunction gives $-K_X = h + a + 2L$.

Description of π_1. The first section cuts out F. Pushing forward the remaining summand along the $\mathbb{P}^2$-factor gives, with vanishing higher direct images,

$$V_3^\vee \otimes \mathcal{U}_1^\vee \otimes (\mathcal{U}_3/\mathcal{U}_1)^\vee,$$

so the induced morphism on F is

$$\varphi \colon V_3 \otimes \mathcal{O}_F \longrightarrow \mathcal{U}_1^\vee \otimes (\mathcal{U}_3/\mathcal{U}_1)^\vee, \qquad x \longmapsto \omega_x(u, -)|_{U_3}.$$

Thus $X = \mathbb{P}_F(\ker \varphi)$, with point fibres at rank two and $\mathbb{P}^1$ fibres at rank one. Over $\mathbb{P}(\Lambda)$ the map factors through $V_3 \otimes \mathcal{O}(-1) \to (V_4/\mathcal{U}_1)^\vee$, a rank-three morphism with determinant in $H^0(\mathbb{P}(\Lambda), \mathcal{O}(2))$. It drops rank precisely along Γ, where its image is $N_u^\perp$. The further quotient to $(U_3/u)^\vee$ has rank at most one exactly when $N_u \subset U_3$. Hence

$$D_1(\varphi) = S = \{(u(v), T_v Q_0) : v \in Q_0\}, \qquad D_0(\varphi) = \varnothing.$$

The smooth conic Γ and the linear condition $N_u \subset U_3$ show that S is a smooth regular codimension-two degeneracy locus. Locally, elementary operations reduce φ to a unit and two functions defining S. The standard blow-up incidence construction (cf. [BFMT, Lemma 3.1]) gives $X \simeq \mathrm{Bl}_S F$. The exceptional divisor has class $E_S = a + L - h$.

Description of π_2. On $B = \mathbb{P}^2 \times \mathrm{Gr}(3, V_4)$ the restriction of the net and the rank-three identity $\bigwedge^2 \mathcal{U}_3^\vee \simeq \mathcal{U}_3 \otimes \det \mathcal{U}_3^\vee$ induce

$$\chi \colon \mathcal{O}_B(-1,-1) \longrightarrow \mathcal{U}_3 \xrightarrow{\lambda} \mathcal{O}_B,$$

where $\lambda \in V_4^\vee$ defines Λ. The vanishing of the composition is exactly the $(1,1)$ equation $p(x) \in U_3$, hence its zero locus is G. On G, a nonzero χ determines the unique line $u = \operatorname{Im}\chi \subset U_3 \cap \Lambda$, whereas its zeros form

$$S' = Z_G(\chi) = \{(x, U_3) : x \in \Delta,\ \ker \omega_x \subset U_3\} = \{(x(v), T_v Q_0) : v \in Q_0\}.$$

This is a smooth surface of codimension two. Since $[\Lambda] \notin Q_0^\vee$, the restriction of λ to $\mathcal{U}_3$ is surjective near S'; there χ is a regular section of its rank-two kernel, with line-incidence X. The same blow-up lemma gives $X \simeq \mathrm{Bl}_{S'} G$, with $E_{S'} = h + L - a$.

Description of π_3. Put $B' = \mathbb{P}^2 \times \mathbb{P}(\Lambda)$ and $\mathcal{U}_1 = \mathcal{O}_{B'}(0,-1)$. The choices of $U_3 \supset u$ form $\mathbb{P}_{B'}((V_4/\mathcal{U}_1)^\vee)$, with the projectivization of lines. Relative pushforward of the second section gives the rank-three bundle $(V_4/\mathcal{U}_1)^\vee(1,1)$, with vanishing higher direct images, and the induced morphism is

$$\sigma \colon \mathcal{O}_{B'}(-1,-1) \xrightarrow{\omega_x(u,-)} (V_4/\mathcal{U}_1)^\vee.$$

The incidence is the locus where the line $U_3^\perp$ contains $\operatorname{Im}\sigma$. Its exceptional centre is

$$C = Z(\sigma) = \{(x,u) : \omega_x(u,-) = 0\} = \{(x, p(x)) : x \in \Delta\}.$$

Since $H^0(\mathbb{P}(\Lambda), \mathcal{U}_1^\vee \otimes (V_4/\mathcal{U}_1)^\vee) \simeq \bigwedge^2 V_4^\vee$, the net induces a general section of the globally generated rank-three bundle $(V_4/\mathcal{U}_1)^\vee(1,1)$. Writing h, a for the hyperplane classes of B', its zero locus has class

$$[C] = c_3\big((V_4/\mathcal{U}_1)^\vee(1,1)\big) = 2h^2a + 2ha^2.$$

Thus C is a smooth regular codimension-three curve, isomorphic to the conic Δ and of bidegree $(2,2)$. The rank-three blow-up incidence construction (cf. [BFMT, Lemma 3.1] locally) yields $X \simeq \mathrm{Bl}_C B'$, with exceptional class $E_C = h + a - L$.

Cones and the remaining fibrations. Since $X = \mathrm{Bl}_C B'$, we have $\rho(X) = 3$. The classes h, a, L are nef, and the lines in the exceptional fibres of π_1, π_2, π_3 have intersection vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, respectively. They therefore generate the Mori cone dually to $\mathrm{Nef}(X) = \langle h, a, L\rangle$; the three blow-ups are the extremal K-negative contractions, as $-K_X = h + a + 2L$. The classes h, a define fibrations over the two $\mathbb{P}^2$ factors, both with general fibre $\mathbb{P}^2$. Finally, the projection $F \to \mathbb{P}(V_4^\vee)$ has fibres $\mathbb{P}^1$ for $U_3 \neq \Lambda$ and $\mathbb{P}^2$ for $U_3 = \Lambda$. The centre S maps isomorphically to $Q_0^\vee$, disjoint from $[\Lambda]$. Blowing up S replaces each $\mathbb{P}^1$ meeting it transversally in one point by two intersecting $\mathbb{P}^1$'s, while the fibre over $[\Lambda]$ stays $\mathbb{P}^2$. This proves the claimed non-flatness. Rationality follows from the blow-up of B'. ◇

References

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