Fano 3-273-60-d. $\mathscr{Z}(\mathbb{P}^2 \times \mathrm{Fl}(2,3,5),\ \det \mathcal{U}_2^\vee \oplus \det \mathcal{U}_2^\vee \oplus \mathcal{Q}_{\mathbb{P}^2} \boxtimes (V_5/\mathcal{U}_3))$.

Invariants. $h^0(-K) = 60,\ (-K)^4 = 273,\ h^{1,1} = 3,\ h^{3,1} = 0,\ h^{2,2} = 5,\ -\chi(T) = -4$.

Description. We regard $\mathrm{Fl}(2,3,V_5) \subset \mathrm{Gr}(2,V_5) \times \mathrm{Gr}(3,V_5)$, so that $X \subset M_1 \times M_2 \times M_3$ with $M_1 = \mathbb{P}^2 = \mathbb{P}(V_3)$, $M_2 = \mathrm{Gr}(2,V_5)$ and $M_3 = \mathrm{Gr}(3,V_5)$. The sections are $\omega_1, \omega_2 \in \bigwedge^2 V_5^\vee$, cutting $W = \mathrm{Gr}(2,V_5) \cap \{\omega_1 = \omega_2 = 0\}$, and an injection $\varphi\colon V_3^\vee \hookrightarrow V_5$ with image A_3. For $x \in \mathbb{P}^2$ put $\Pi_x = \varphi(x^\perp) \subset A_3$. Let U_2^0 be the unique plane of A_3 lying in W, and x_0 the point with $\Pi_{x_0} = U_2^0$. Let

$$G = \{(x, U_3) : U_3 \supset \Pi_x\} = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)) \subset \mathbb{P}^2 \times \mathrm{Gr}(3,V_5).$$

(1) $\pi_1\colon X \to \mathrm{Fl}(2,3,V_5)$ is a K-negative extremal small contraction. It contracts the plane $P = \mathbb{P}^2 \times \{(U_2^0, A_3)\}$, with normal bundle $\mathcal{O}(-1)^{\oplus 2}$, to a point where its image is locally the cone over $\mathbb{P}^2 \times \mathbb{P}^1$.
(2) $\pi_2\colon X \to G \subset \mathbb{P}^2 \times \mathrm{Gr}(3,V_5)$ is the blow up of the Fano fourfold G of type 2-513-105-b along a surface $T \simeq \mathbb{F}_1$; it is a K-negative extremal divisorial contraction.
(3) $\pi_3\colon X \to \mathbb{P}^2 \times W$ is a K-negative extremal small contraction onto a fourfold. It contracts the plane $P' = \{x_0\} \times \{U_2^0\} \times \{U_3 \supset U_2^0\}$, with normal bundle $\mathcal{O}(-1)^{\oplus 2}$, to a point where its image is locally the cone over $\mathbb{P}^2 \times \mathbb{P}^1$.

- Notation: $h = \mathcal{O}_{\mathbb{P}^2}(1)$, $L_1 = \det \mathcal{U}_2^\vee$, $L_2 = \det \mathcal{U}_3^\vee$ restricted to X; then $-K_X = h + L_1 + L_2$.
- $\mathrm{Nef}(X) = \langle h, L_1, L_2 \rangle$ and $\overline{\mathrm{NE}}(X) = \langle c_h, c_1, c_2 \rangle$ (dual bases), both simplicial; the three rays are contracted by π_1, π_2, π_3.
- The face $h^\perp$ gives a del Pezzo fibration $X \to \mathbb{P}^2$ of degree 8, with general fiber $\mathrm{Bl}_1 \mathbb{P}^2$ and a reducible fiber over x_0.
- Rationality: yes.

Semiorthogonal decompositions.

- A full exceptional collection of length $9 + 4 = 13$ from $X = \mathrm{Bl}_T G$.

Explanation.

Setup. On $\mathbb{P}^2$ let $\mathcal{Q}_{\mathbb{P}^2} = V_3/\mathcal{O}(-1)$, and on $\mathrm{Gr}(3,V_5)$ write $\mathcal{Q} = V_5/\mathcal{U}_3$. We have $H^0(\mathbb{P}^2 \times \mathrm{Fl}(2,3,5), \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{Q}) = V_3 \otimes V_5 \ni \varphi$, and the section vanishes at (x, U_3) if and only if its image in $(V_3/x) \otimes (V_5/U_3)$ vanishes. Equivalently, the composite

$$x^\perp \hookrightarrow V_3^\vee \xrightarrow{\varphi} V_5 \to V_5/U_3$$

is zero, i.e. $\Pi_x \subset U_3$. The two sections of L_1 are Plücker forms on U_2. Thus

$$X = \{(x, U_2 \subset U_3) :\ U_2 \in W,\ \Pi_x \subset U_3\}.$$

Since $\Pi_x = \varphi(x^\perp)$ and $x^\perp \simeq \Omega_{\mathbb{P}^2}(1)$, we have $A_3/\Pi \simeq \mathcal{O}(1)$ and $V_5/\Pi \simeq \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)$, the extension splitting because $H^1(\mathbb{P}^2, \mathcal{O}(1)) = 0$. Hence $G = \mathbb{P}_{\mathbb{P}^2}(V_5/\Pi)$, with $L_2|_G = \xi + h$ (ξ the relative $\mathcal{O}(1)$), $-K_G = 3L_2 + h$ and $(-K_G)^4 = 513$. Identifying $\mathrm{Gr}(3, V_5) \simeq \mathrm{Gr}(2, V_5^\vee)$ by $U_3 \mapsto U_3^\perp$, the condition $\Pi_x \subset U_3$ becomes $\varphi^\vee(U_3^\perp) \subset x$, so G is the zero locus of a general section of $\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee$ on $\mathbb{P}^2 \times \mathrm{Gr}(2,5)$, a fourfold of type 2-513-105-b (equivalently, the zero locus of $\mathcal{U}_2^\vee \oplus \mathcal{U}_2^\vee$ in $\mathrm{Fl}(2,3,5)$). Its second extremal contraction $G \to \mathrm{Gr}(3, V_5)$ is small: it contracts the section $\{U_3 = A_3\} \simeq \mathbb{P}^2$, with normal bundle $\mathcal{O}(-1)^{\oplus 2}$, to a point. This distinguishes G from $\mathrm{Bl}_{\mathrm{line}}\mathbb{P}^4 = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 2})$, which has the same invariants $(\rho, (-K)^4, h^0(-K), h^{2,2}) = (2, 513, 105, 3)$ but a divisorial second contraction. On $\mathrm{Gr}(2, A_3) \simeq \mathbb{P}^2$ the two Plücker forms cut two general lines, which meet in the single point U_2^0.

Description of π_1. The projection π_1 forgets $x \in M_1$. By Bott vanishing, $\pi_{1*}(\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{Q}) = V_3 \otimes \mathcal{Q}$ with $R^{>0}\pi_{1*} = 0$. Over $\mathrm{Fl}_W = \{U_2 \subset U_3 : U_2 \in W\} = \mathbb{P}_W(V_5/\mathcal{U}_2)$, a smooth sixfold, the section becomes

$$\bar\varphi\colon V_3^\vee \otimes \mathcal{O} \xrightarrow{\varphi} V_5 \otimes \mathcal{O} \to \mathcal{Q},$$

of rank $3 \to 2$. The fiber over (U_2, U_3) is $\{x : \bar\varphi(x^\perp) = 0\}$: empty if $\operatorname{rank}\bar\varphi = 2$, one point if $\operatorname{rank}\bar\varphi = 1$, and the whole $\mathbb{P}^2$ if $\bar\varphi = 0$. Hence $\pi_1(X) = D_1(\bar\varphi)$, of expected codimension $(3-1)(2-1) = 2$, and $D_0(\bar\varphi) = \{U_3 = A_3,\ U_2 \subset A_3,\ U_2 \in W\} = \{(U_2^0, A_3)\}$, of expected codimension 6. Since φ is a general element of $\mathrm{Hom}(V_3^\vee, V_5) \subset H^0(\mathrm{Hom}(V_3^\vee \otimes \mathcal{O}, \mathcal{Q}))$ and $\mathcal{Q}$ is globally generated, these loci have the expected codimension, $D_1 \setminus D_0$ is smooth, and D_0 is a reduced point [Har77, III.10.7]. Near it, the six entries of the 2×3 matrix $\bar\varphi$ are local coordinates of Fl_W, $\pi_1(X) = \{\operatorname{rank} \le 1\}$ is the cone over $\mathbb{P}^2 \times \mathbb{P}^1$, and

$$X = \{(\bar\varphi, x) : \bar\varphi|_{x^\perp} = 0\} = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}\big).$$

So π_1 is an isomorphism outside P and contracts $P \simeq \mathbb{P}^2$, with $N_{P/X} = \mathcal{O}(-1)^{\oplus 2}$. For a line $c_h \subset P$ we have $h \cdot c_h = 1$, $L_1 \cdot c_h = L_2 \cdot c_h = 0$ and $-K_X \cdot c_h = 1$.

Description of π_2. The projection π_2 forgets U_2. The rank-4 part of the bundle is pulled back from $M_1 \times M_3$, and its zero locus there is G, since the section vanishes exactly when $\Pi_x \subset U_3$. The two sections of L_1 live on the $\mathbb{P}^2$-bundle $\mathrm{Gr}(2, \mathcal{U}_3) = \mathbb{P}_G(\bigwedge^2 \mathcal{U}_3)$ over G, so π_2 maps X to G. The fiber of π_2 over (x, U_3) is $\{U_2 \subset U_3 : \omega_1(U_2) = \omega_2(U_2) = 0\}$, a plane $\mathrm{Gr}(2, U_3) = \mathbb{P}(\bigwedge^2 U_3)$ cut by two hyperplanes. Since π_{2*} of the Plücker bundle of this $\mathbb{P}^2$-bundle is $\bigwedge^2 \mathcal{U}_3^\vee$, the two sections give

$$\tau = (\omega_1|_{U_3}, \omega_2|_{U_3})\colon \bigwedge^2 \mathcal{U}_3 \longrightarrow \mathcal{O}_G^{\oplus 2},$$

and $X = \mathbb{P}(\ker\tau) \subset \mathbb{P}_G(\bigwedge^2 \mathcal{U}_3)$. The map τ is general, because ω_1, ω_2 are general in $\bigwedge^2 V_5^\vee$, which generates $\bigwedge^2 \mathcal{U}_3^\vee$. So

$$T := D_1(\tau) \text{ has codimension 2}, \qquad D_0(\tau) = \emptyset \text{ (codimension 6)},$$

π_2 is an isomorphism over $G \setminus T$, and its fiber over T is a line, of class c_1, with $L_1 \cdot c_1 = 1$ and $h \cdot c_1 = L_2 \cdot c_1 = 0$. Since $\rho(X/G) = 1$, $L_2 + h$ is nef with $(L_2+h)^\perp \cap \overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0} c_1$, and all fibers are at most one-dimensional, this gives

$$X = \mathrm{Bl}_T G, \qquad -K_X = \pi_2^*(-K_G) - E_1, \qquad E_1 = 2L_2 - L_1.$$

The blow up formulas give $h^2 \cdot T = -h^2 E_1^2 = 1$, $(hL_2)_T = 3$ and $L_2^2 \cdot T = 5$; moreover $b_2(T) = h^{2,2}(X) - b_4(G) = 5 - 3 = 2$ and $q(T) = p_g(T) = 0$. Hence T is rational with $K_T^2 = 8$, and $T \to \mathbb{P}^2$, defined by the base point free class $h|_T$, is birational. Thus $T \simeq \mathbb{F}_1 = \mathrm{Bl}_{x_0} \mathbb{P}^2$ with

$$h|_T = \ell, \qquad L_2|_T = 3\ell - 2e, \qquad c_1(N_{T/G}) = K_T + (3L_2+h)|_T = 7\ell - 5e, \qquad c_2(N_{T/G}) = 8.$$

These values reproduce $E_1^3 h = -7$, $E_1^3 L_2 = -11$ and $E_1^4 = c_2 - c_1^2 = -16$ from the original model. The exceptional curve e is the conic $C_0 \subset G_{x_0}$ of planes $U_3 \supset U_2^0$ on which $\omega_1|_{U_3}$ and $\omega_2|_{U_3}$, which both vanish on U_2^0, become proportional.

Since $\rho(X/G) = 1$ and $L_2 + h$ is nef with $(L_2+h)^\perp \cap \overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0} c_1$, π_2 is the K-negative extremal divisorial contraction of $\mathbb{R}_{\geq 0} c_1$.

Description of π_3. The projection π_3 forgets U_3. The two sections of L_1 are pulled back from M_2 and cut out W, and π_3 maps X to $\mathbb{P}^2 \times W$. Pushing forward along the projective bundle $\{U_3 \supset U_2\} = \mathbb{P}(V_5/\mathcal{U}_2)$, the condition $\Pi_x \subset U_3$ becomes the composite

$$\mu \colon \Pi \hookrightarrow V_5 \otimes \mathcal{O} \longrightarrow V_5/\mathcal{U}_2$$

on $\mathbb{P}^2 \times W$, from rank 2 to rank 3. The fiber over (x, U_2) is the set of $U_3 \supset U_2 + \Pi_x$, that is, of lines in V_5/U_2 containing $\operatorname{Im} \mu_{(x,U_2)}$. It is empty if $\operatorname{rank} \mu = 2$, a point if $\operatorname{rank} \mu = 1$, and a plane $\mathbb{P}(V_5/U_2)$ if $\mu = 0$, i.e. $U_2 = \Pi_x$, which happens only at (x_0, U_2^0). So $\pi_3(X) = D_1(\mu)$, a fourfold of expected codimension $(2-1)(3-1) = 2$, and $D_0(\mu) = \{(x_0, U_2^0)\}$. The map μ is general, since $H^0(\mathbb{P}^2 \times W, \mathrm{Hom}(\Pi, V_5/\mathcal{U}_2)) \supset V_3 \otimes V_5 \ni \varphi$ generates this bundle; hence $D_0(\mu)$ is a reduced point and $D_1 \setminus D_0$ is smooth. Near D_0 the six entries of μ are local coordinates, the image is the cone over $\mathbb{P}^2 \times \mathbb{P}^1$, and

$$X = \{(\mu, \ell \in \mathbb{P}^2) : \operatorname{Im} \mu \subset \ell\} = \mathrm{Tot}\big(\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}\big).$$

So π_3 is an isomorphism outside P' and contracts $P' \simeq \mathbb{P}^2$, with $N_{P'/X} = \mathcal{O}(-1)^{\oplus 2}$. For a line $c_2 \subset P'$ we have $L_2 \cdot c_2 = 1$, $h \cdot c_2 = L_1 \cdot c_2 = 0$ and $-K_X \cdot c_2 = 1$; so π_3 is the K-negative extremal small contraction of $\mathbb{R}_{\geq 0} c_2$, supported by $h + L_1$.

Extremal contractions and cones. The three curves c_h, c_1, c_2 are dual to the three base point free classes h, L_1, L_2, which form a basis of $N^1(X)$ since $\rho(X) = 3$. Therefore

$$\overline{\mathrm{NE}}(X) = \langle c_h, c_1, c_2 \rangle, \qquad \mathrm{Nef}(X) = \langle h, L_1, L_2 \rangle,$$

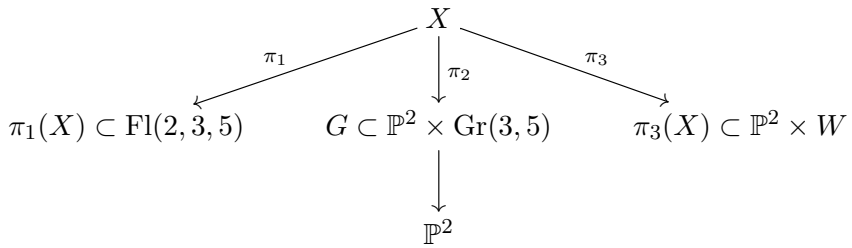
both simplicial, with

	c_h	c_1	c_2
h	1	0	0
L_1	0	1	0
L_2	0	0	1
E_1	0	-1	2
$-K_X$	1	1	1

The three extremal contractions are the three projections: π_1 contracts $\mathbb{R}_{\geq 0}c_h$ (supported by L_1+L_2), π_2 contracts $\mathbb{R}_{\geq 0}c_1$ (supported by $h+L_2$), and π_3 contracts $\mathbb{R}_{\geq 0}c_2$ (supported by $h+L_1$). The planes P and P' meet in the single point (x_0, U_2^0, A_3), and $E_1 \cdot c_2 = 2$: E_1 meets P' along a conic.

The two-dimensional faces give the following contractions.

- The face $h^\perp = \langle c_1, c_2\rangle$ gives the composite $X \to G \to \mathbb{P}^2$. For $x \neq x_0$ the fiber $G_x \simeq \mathbb{P}^2$ meets T in one point, so the fiber of X is $\mathrm{Bl}_1 G_x$, a del Pezzo surface of degree 8 mapped by $L_1 = 2\ell - e$ onto the cubic scroll $W \cap \{U_2 \cap \Pi_x \neq 0\}$ (in agreement with $h^2L_1^2 = 3$). Over x_0 the fiber is $P' \cup \pi_2^{-1}(C_0)$, the two components meeting along the conic over C_0.
- The face $L_2^\perp$ gives $X \to \Sigma = \{U_3 : \dim(U_3 \cap A_3) \geq 2\} \subset \mathrm{Gr}(3, V_5)$. It is birational and contracts E_1 and P.
- The face $L_1^\perp$ gives $X \to D = W \cap \{U_2 \cap A_3 \neq 0\}$. It is a fibration in conics of class $c_2 + c_h$ over a singular threefold of degree 5, as $L_1^4 = 0$ and $L_1^3 h = 5$.



Derived category. By Orlov's formulas [Orl92], $\mathrm{D^b}(X) = \langle \mathrm{D^b}(G), \mathrm{D^b}(T)\rangle$; the $\mathbb{P}^2$-bundle G over $\mathbb{P}^2$ has a full exceptional collection of length 9, and $T \simeq \mathbb{F}_1$ one of length 4. This gives $13 = \chi_{\mathrm{top}}(X)$ exceptional objects.

◇

References

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