Fano 3-289-63-a. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Fl}(2,4,6),\ (\mathcal{U}_2^\vee)^{\oplus 2} \oplus (\mathcal{U}_4/\mathcal{U}_2)^\vee \otimes \det \mathcal{U}_2^\vee \oplus \mathcal{Q}_{\mathbb{P}^2} \boxtimes V_6/\mathcal{U}_4\big)$, with $\mathrm{Fl}(2,4,6) \subset \mathrm{Gr}(2,6) \times \mathrm{Gr}(4,6)$.

Invariants. $h^0(-K) = 63,\ (-K)^4 = 289,\ h^{1,1} = 3,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 6,\ -\chi(T) = -4$.

Description.

(1) $\pi_1 \colon X \to \mathrm{Gr}(2,6) \times \mathrm{Gr}(4,6)$ is an extremal small contraction: it contracts a plane with normal bundle $\mathcal{O}(-1)^{\oplus 2}$ to a point, and is an isomorphism elsewhere.
(2) $\pi_2 \colon X \to \mathbb{P}^2 \times \mathrm{Gr}(4,6)$ is an extremal divisorial contraction onto a special member Y of the family 2-352-75-b, the blow-up of Y along a surface $\mathbb{P}^1 \times \mathbb{P}^1$.
(3) $\pi_3 \colon X \to \mathbb{P}^2 \times \mathrm{Gr}(2,6)$ is an extremal divisorial contraction onto the blow-up of a quadric fourfold Q^4 along a smooth conic (2-352-75-d), the blow-up of it along a surface $\mathbb{F}_1$.

- h, L_2, L_4 denote the pullbacks of the hyperplane class of $\mathbb{P}^2$ and of the Plücker classes of $\mathrm{Gr}(2,6)$ and $\mathrm{Gr}(4,6)$.
- $\mathrm{Nef}(X) = \langle h, L_2, L_4 \rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_3, [Orl] and [Kap], $\mathrm{D}^{\mathrm{b}}(X)$ has a full exceptional collection of length $4+2+2+6=14$.

Explanation.

Write $\mathbb{P}^2 = \mathbb{P}(V_3)$ and V_6 for the space of $\mathrm{Fl}(2,4,6)$, so that X consists of triples (z, U_2, U_4) with $U_2 \subset U_4$. The two sections of $\mathcal{U}_2^\vee$ are linear forms λ_1, λ_2 with common kernel K_4, and impose $U_2 \subset K_4$. The section of $(\mathcal{U}_4/\mathcal{U}_2)^\vee \otimes \det \mathcal{U}_2^\vee$ is a general 3-form σ, and imposes $\sigma(U_2, U_2, U_4) = 0$. The section of $\mathcal{Q}_{\mathbb{P}^2} \boxtimes V_6/\mathcal{U}_4$ is a general injection $\varphi \colon V_3^\vee \to V_6$, with image Φ, and imposes $\Phi_z := \varphi(z^\perp) \subset U_4$. Let φ_0 span the line $\Phi \cap K_4$, and let $\ell = \{z : \varphi_0 \in \Phi_z\}$, a line of $\mathbb{P}^2$. Since $\bigwedge^3 K_4^\vee \simeq K_4 \otimes \det K_4^\vee$, the restriction $\sigma|_{K_4}$ is given by a vector $\kappa \in K_4$, and σ vanishes on a 3-space $K_3 \subset K_4$ if and only if $\kappa \in K_3$.

If $U_4 \cap K_4 = U_2$ has dimension 2, choose $w_1, w_2 \in U_4$ with $\lambda_i(w_j) = \delta_{ij}$: then $(\sigma \wedge \lambda_1)|_{U_4}$ and $(\sigma \wedge \lambda_2)|_{U_4}$ are, up to sign, $\sigma(U_2, U_2, w_2)$ and $\sigma(U_2, U_2, w_1)$. Hence there the condition $\sigma(U_2, U_2, U_4) = 0$ is equivalent to the vanishing of the two Plücker linear forms $\sigma \wedge \lambda_1, \sigma \wedge \lambda_2 \in \bigwedge^4 V_6^\vee$ on U_4.

Description of π_1. The fiber of π_1 over (U_2, U_4) is the zero locus in $\mathbb{P}^2$ of the composite $z^\perp \otimes \mathcal{O} \to V_3^\vee \otimes \mathcal{O} \xrightarrow{\varphi} V_6/U_4 \otimes \mathcal{O}$, namely $\{z : \Phi_z \subset U_4\}$: it is empty if $\Phi \to V_6/U_4$ has rank 2, a reduced point if it has rank 1, and $\mathbb{P}^2$ if $\Phi \subset U_4$. On the plane $\{U_4 \supset \Phi\} \simeq \mathbb{P}(V_6/\Phi)$ the Plücker class has degree one, so the two forms $\sigma \wedge \lambda_i$ vanish at a single U_4^0, and then $U_2^0 = U_4^0 \cap K_4$. Hence π_1 contracts exactly the plane $\Pi = \mathbb{P}^2 \times \{(U_2^0, U_4^0)\}$, and is an isomorphism elsewhere. Its image is cut out, in the smooth sixfold of pairs $U_2 \subset U_4$ with $U_2 \subset K_4$ and $\sigma(U_2, U_2, U_4) = 0$, by the determinantal condition $\dim(U_4 \cap \Phi) \geq 2$ of the expected codimension 2; it is Cohen–Macaulay and smooth outside one point, hence normal. A line $f_1 \subset \Pi$ has $h \cdot f_1 = 1$ and $L_2 \cdot f_1 = L_4 \cdot f_1 = 0$, and the nef class $L_2 + L_4$ vanishes exactly on the curves of Π; so π_1 is the contraction of the extremal ray spanned by f_1, which is K_X-negative since X is Fano. By [Kaw], $N_{\Pi/X} \simeq \mathcal{O}(-1)^{\oplus 2}$.

Description of π_2. By [BFMT, Proposition 3.3], $G = \{(z, U_4) : \Phi_z \subset U_4\} = \mathscr{Z}(\mathbb{P}^2 \times \mathrm{Gr}(4,6), \mathcal{Q}_{\mathbb{P}^2} \boxtimes V_6/\mathcal{U}_4)$ is a $\mathrm{Gr}(2,4)$-bundle over $\mathbb{P}^2$, and it contains the image of π_2. The fiber over $(z, U_4) \in G$ consists of the $U_2 \subset U_4 \cap K_4$ with $\sigma(U_2, U_2, U_4) = 0$. Where $\dim(U_4 \cap K_4) = 2$, U_2 is determined and the condition is $(\sigma \wedge \lambda_i)|_{U_4} = 0$; so

$$\pi_2(X) = Y = G \cap H_{\sigma\wedge\lambda_1} \cap H_{\sigma\wedge\lambda_2} = \mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(4,6), \mathcal{Q}_{\mathbb{P}^2} \boxtimes V_6/\mathcal{U}_4 \oplus \mathcal{O}(L_4)^{\oplus 2}\big),$$

the model of 2-352-75-b, for a special section. Where $K_3 = U_4 \cap K_4$ has dimension 3, the condition on $U_2 \subset K_3$ contains $\sigma(U_2, U_2, K_3) = 0$, that is $\sigma|_{K_3} = 0$, or $\kappa \in K_3$; when this holds, the remaining condition $\sigma(U_2, U_2, U_4) = 0$ is one linear condition on $U_2 \in \mathbb{P}(K_3^\vee)$, and U_2 runs over a line. Moreover $U_4 \supset K_3 + \Phi_z$ forces $K_3 \cap \Phi_z \neq 0$, that is, $\varphi_0 \in K_3$ and $z \in \ell$. Hence the fibers of π_2 are points, except for lines over the surface

$$T_2 = \{(z, K_3 + \Phi_z) : \ z \in \ell, \ \langle \kappa, \varphi_0 \rangle \subset K_3 \subset K_4\} \simeq \ell \times \mathbb{P}(K_4/\langle \kappa, \varphi_0 \rangle) \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

These lines f_2 have $L_2 \cdot f_2 = 1$ and $h \cdot f_2 = L_4 \cdot f_2 = 0$, and the nef class $h + L_4$ vanishes exactly on them; so π_2 is the contraction of a K_X-negative extremal ray with one-dimensional fibers, and by [And, Thm. 2.3], Y is smooth and $X \simeq \mathrm{Bl}_{T_2} Y$.

Description of π_3. The image of π_3 lies in $\mathbb{P}^2 \times \mathrm{Gr}(2, K_4) = \mathbb{P}^2 \times Q^4$, where $Q^4 \subset \mathbb{P}(\bigwedge^2 K_4)$. For $U_2 \in Q^4$, the condition on $U_4 \supset U_2 + \Phi_z$ is $U_4 \subset \ker \sigma(U_2, U_2, \cdot)$, which requires that the linear form $\sigma(U_2, U_2, \cdot)|_\Phi \in \Phi^\vee \simeq V_3$ lies in z. This is the vanishing of the composite

$$\textstyle\bigwedge^2 U_2 \otimes \mathcal{O} \longrightarrow \bigwedge^2 K_4 \otimes \mathcal{O} \xrightarrow{\sigma} \Phi^\vee \otimes \mathcal{O} \simeq V_3 \otimes \mathcal{O} \longrightarrow \mathcal{Q}_{\mathbb{P}^2},$$

so the image is $Z = \mathscr{Z}(\mathbb{P}^2 \times Q^4, \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{O}(1))$. By [BFMT, Lemma 3.1], Z is the blow-up of Q^4 along the conic $C_3 = Q^4 \cap \mathbb{P}(\ker(\bigwedge^2 K_4 \to \Phi^\vee))$, smooth for general σ; this is the model of 2-352-75-d. Over $(z, U_2) \in Z$, $U_4 = U_2 + \Phi_z$ is unique when $U_2 \cap \Phi_z = 0$. Otherwise $\varphi_0 \in U_2 \cap \Phi_z$, so $z \in \ell$ and $U_2 = \langle \varphi_0, u \rangle$, and $(z, U_2) \in Z$ means $\sigma(\varphi_0, u, \Phi_z) = 0$; then U_4 runs over the line of 4-spaces between $U_2 + \Phi_z$ and $\ker \sigma(U_2, U_2, \cdot)$. The locus

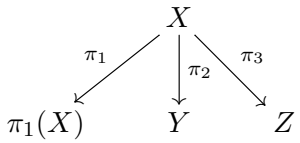
$$S_3 = \{(z, \langle \varphi_0, u \rangle) : \ z \in \ell, \ \sigma(\varphi_0, u, \Phi_z) = 0\}$$

is a divisor of bidegree $(1,1)$ in $\ell \times \mathbb{P}(K_4/\varphi_0) \simeq \mathbb{P}^1 \times \mathbb{P}^2$, isomorphic to $\mathbb{F}_1$. The fibers f_3 over it have $L_4 \cdot f_3 = 1$ and $h \cdot f_3 = L_2 \cdot f_3 = 0$, and the nef class $h + L_2$ vanishes exactly on them. As before, [And, Thm. 2.3] gives $X \simeq \mathrm{Bl}_{S_3} Z$. In particular X is rational.

Cones. The classes h, L_2, L_4 are nef, and the curves f_1, f_2, f_3 form the dual basis:

	f_1	f_2	f_3
h	1	0	0
L_2	0	1	0
L_4	0	0	1

Hence $\mathrm{Nef}(X) = \langle h, L_2, L_4 \rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2, f_3 \rangle$ are dual simplicial cones.



◇

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