Fano 3-305-66-b. $\mathscr{Z}\big(\mathrm{Fl}(2,3,5)\times\mathrm{Gr}(2,4),\mathcal{O}_{\mathrm{Gr}(2,4)}(1)\oplus\mathcal{U}_2^\vee\oplus\det\mathcal{U}_2^\vee\oplus(V_5/\mathcal{U}_3)\boxtimes\mathcal{U}^\vee\big)$, where $\mathcal{U}_2\subset\mathcal{U}_3\subset V_5\otimes\mathcal{O}$ are the tautological bundles on the flag variety and $\mathcal{U}$ is the rank-two tautological bundle on $\mathrm{Gr}(2,4)$.

Invariants. $h^0(-K)=66,\quad (-K)^4=305,\quad h^{1,1}=3,\quad h^{2,2}=5,\quad -\chi(T)=-6.$

Description. We regard the flag variety as an incidence locus in the ordered product $\mathrm{Gr}(2,V_5)\times\mathrm{Gr}(3,V_5)\times\mathrm{Gr}(2,W_4)$, and π_i omits its i-th factor. Write b,L,h for the respective Plücker classes restricted to X. Let $Q^3_U\subset\mathrm{Gr}(2,V_5)$ and $Q^3_P\subset\mathrm{Gr}(2,W_4)$ be the smooth quadric threefolds cut out by the linear and alternating equations of the respective first and third factors. There is a smooth del Pezzo fourfold $W_5\subset\mathrm{Gr}(3,V_5)$ of degree five, of type 1-405-85-a, containing two transverse planes Π_K,Π_Φ meeting in one point.

(1) $\pi_1\colon X\to Y_1$ is an elementary extremal divisorial contraction with $Y_1\simeq\mathrm{Bl}_{\Pi_\Phi}W_5$, of type 2-352-75-c, and $X\simeq\mathrm{Bl}_{S_1}Y_1$, where $S_1\simeq\mathrm{Bl}_p\mathbb{P}^2\simeq\mathbb{F}_1$ is the strict transform of Π_K.
(2) $\pi_2\colon X\to Z\subset Q^3_U\times Q^3_P$ is an elementary extremal small contraction onto a determinantal fourfold. It contracts a unique plane $\Pi_0\simeq\mathbb{P}^2$, with $N_{\Pi_0/X}\simeq\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus2}$, to the unique singular point of Z.
(3) $\pi_3\colon X\to Y_3$ is an elementary extremal divisorial contraction with $Y_3\simeq\mathrm{Bl}_{\Pi_K}W_5$, again of type 2-352-75-c, and $X\simeq\mathrm{Bl}_{S_3}Y_3$, where $S_3\simeq\mathbb{F}_1$ is the strict transform of Π_Φ.

- $X\simeq\mathrm{Bl}_{\Pi_K\cup\Pi_\Phi}W_5$; the fiber over $\Pi_K\cap\Pi_\Phi$ is $\mathbb{P}^1\times\mathbb{P}^1$.
- $\mathrm{Nef}(X)=\langle b,L,h\rangle$ and $\overline{\mathrm{NE}}(X)=\langle f_1,f_2,f_3\rangle$ are simplicial; the classes f_i are dual to b,L,h and are contracted by π_i.
- X is rational.

Semiorthogonal decompositions. The successive codimension-two blow-ups and Orlov's formula [2] give $\mathrm{D^b}(X)=\langle\mathrm{D^b}(\mathbb{F}_1),\mathrm{D^b}(\mathbb{P}^2),\mathrm{D^b}(W_5)\rangle$. Writing $q:Y_1\to W_5$ for the first blow-up, the embeddings are respectively $j_{1*}(p_1^*(-)\otimes\mathcal{O}_{E_1}(-1))$, $\pi_1^*j_{\Phi*}(p_\Phi^*(-)\otimes\mathcal{O}_{F_\Phi}(-1))$, and $(q\circ\pi_1)^*$, with the standard exceptional inclusions and projections.

Explanation. A general section of $\mathcal{U}_2^\vee$ determines a hyperplane $K_4=\ker\lambda\subset V_5$, and a general section of $\det\mathcal{U}_2^\vee$ restricts to a nondegenerate alternating form ν on K_4. The section of the first summand defines the smooth quadric $Q^3_P=\mathrm{LG}(2,W_4,\omega)\subset\mathrm{Gr}(2,W_4)$, while that of the last summand is determined by an injective map $\varphi:W_4\to V_5$ with image Φ_4. Put $K_3=K_4\cap\Phi_4$ and let $\mu\in V_5^\vee$ have kernel Φ_4. We also write ω for the induced symplectic form on Φ_4. Then

$$X=\{(U_2\subset U_3,P):U_2\subset K_4,\ \nu|_{U_2}=0,\ P\in Q^3_P,\ \varphi(P)\subset U_3\}.$$

Set $Q^3_U=\mathrm{LG}(2,K_4,\nu)$. Adjunction on the original zero locus yields $-K_X=b+L+h$.

The composite restriction morphisms of alternating forms define

$$\bigwedge^3\mathcal{U}_3\longrightarrow\bigwedge^3V_5\otimes\mathcal{O}\xrightarrow{(\lambda\wedge\nu,\ \mu\wedge\omega)}\mathcal{O}^{\oplus2}.$$

Their zero locus is the smooth intersection

$$W_5=\mathscr{Z}_{\mathrm{Gr}(3,V_5)}\big(\mathcal{O}(1)^{\oplus2}\big),$$

a del Pezzo fourfold of degree five. The two distinguished planes are

$$\Pi_K=W_5\cap\mathrm{Gr}(3,K_4),\qquad\Pi_\Phi=W_5\cap\mathrm{Gr}(3,\Phi_4).$$

The restrictions of the other Plücker equation make each a plane, and their intersection is the point $[K_3]$. At this point their tangent spaces are complementary two-planes in $T_{[K_3]}W_5$: they arise from the independent directions K_4/K_3 and Φ_4/K_3 and the two independent isotropy conditions.

Description of π_1. On Q^3_P the inclusion $\varphi(\mathcal{U}) \subset V_5 \otimes \mathcal{O}$ defines the rank-three quotient through

$$0 \longrightarrow \mathcal{U} \xrightarrow{\varphi} V_5 \otimes \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow 0.$$

The incidence $\varphi(P) \subset U_3$ is $\mathbb{P}_{Q^3_P}(\mathcal{E})$, and the Plücker bundle on $\mathrm{Gr}(3, V_5)$ restricts to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes p^*\mathcal{O}_{Q^3_P}(1)$. Consequently

$$p_*\big(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes p^*\mathcal{O}(1)\big) = \mathcal{E}^\vee(1), \qquad R^i p_* = 0 \quad (i > 0).$$

The equation $\lambda \wedge \nu$ induces the section of $\mathcal{E}^\vee(1)$ given explicitly by the composition

$$\mathcal{E} \otimes \det \mathcal{U} \longrightarrow \bigwedge^3 V_5 \otimes \mathcal{O} \xrightarrow{\lambda\wedge\nu} \mathcal{O},$$

and its zero locus in $\mathbb{P}(\mathcal{E})$ is Y_1. Identifying W_4 with Φ_4, the same incidence is a zero locus on $\mathrm{Fl}(2, 3, V_5)$:

$$Y_1 \simeq \mathscr{Z}\big(\mathrm{Fl}(2, 3, V_5), \mathcal{U}_2^\vee \oplus \det \mathcal{U}_2^\vee \oplus \det \mathcal{U}_3^\vee\big).$$

Thus its homogeneous model is that of 2-352-75-c.

Projection from this incidence to W_5 is the blow-up of Π_Φ. Indeed, away from $\mathrm{Gr}(3, \Phi_4)$ the plane $\varphi(P) = U_3 \cap \Phi_4$ is uniquely determined; above Π_Φ the isotropic planes in U_3 form a $\mathbb{P}^1$. Indeed, in a local Plücker chart where a coefficient of $\omega|_{U_3}$ is nonzero, the equation $(\mu \wedge \omega)|_{U_3} = 0$ eliminates one of the three components of $\mu|_{U_3}$. The other two generate the regular ideal of Π_Φ, and the incidence is the associated relative linear equation. The codimension-two Cayley blow-up lemma [1, Lemma 3.2] therefore applies. Similarly, forgetting U_2 in $X \to Y_1$ replaces U_2 by the uniquely determined $U_3 \cap K_4$ off the strict transform of Π_K and by a $\mathbb{P}^1$ over it. Since Π_K and Π_Φ are transverse in W_5, this strict transform is $S_1 \simeq \mathrm{Bl}_{[K_3]} \Pi_K \simeq \mathbb{F}_1$, smooth of codimension two in Y_1. The same Cayley blow-up description gives $X \simeq \mathrm{Bl}_{S_1} Y_1$. Writing E_1 for its exceptional divisor, the canonical bundle formula, together with $-K_{Y_1} = 2L + h$, gives $E_1 = L - b$.

Description of π_2. On $Q^3_U \times Q^3_P$ consider the universal morphism

$$\psi : \mathcal{U}_U \oplus \mathcal{U}_P \xrightarrow{(\iota,\varphi)} V_5 \otimes \mathcal{O}, \qquad \mathrm{rk}\,\psi : (4 \longrightarrow 5).$$

Its degeneracy scheme $Z = D_3(\psi)$ is the image of π_2, because $\mathrm{rk}\,\psi \le 3$ is precisely the condition that $U_2 + \varphi(P)$ be contained in some three-plane U_3. For rank three this U_3 is unique. The expected codimension is two, and Thom–Porteous gives

$$[Z] = c_2\big(V_5 - \mathcal{U}_U - \mathcal{U}_P\big) = \tfrac{1}{2}b^2 + bh + \tfrac{1}{2}h^2.$$

Here the Chern identities follow from $c(\mathcal{U}_U)c(\mathcal{U}_U^\vee) = c(\mathcal{U}_P)c(\mathcal{U}_P^\vee) = 1$ on the two Lagrangian Grassmannians.

The rank-two locus is

$$D_2(\psi) = \{U_2 = \varphi(P) \subset K_3 : \nu|_{U_2} = \omega|_{U_2} = 0\}.$$

Inside $\mathrm{Gr}(2, K_3) \simeq \mathbb{P}^2$, the two isotropy equations are independent lines, so $D_2(\psi)$ is one reduced point (A, P_0). The Porteous class confirms its scheme-theoretic length: writing $c_i = c_i(V_5 - \mathcal{U}_U - \mathcal{U}_P)$,

$$[D_2(\psi)] = c_3^2 - c_2c_4 = \frac{1}{4}b^3h^3, \qquad \int_{Q^3_U \times Q^3_P} [D_2(\psi)] = 1.$$

At (A, P_0) the two three-dimensional tangent spaces $T_A Q^3_U, T_{P_0} Q^3_P \subset T_A \,\mathrm{Gr}(2, V_5)$ are complementary: their intersection corresponds to the tangent intersection of the two distinct isotropy lines in $\mathrm{Gr}(2, K_3)$. Consequently Z is smooth away from (A, P_0), while its germ there is the rank-at-most-one locus of a 2×3 matrix, the cone over $\mathbb{P}^1 \times \mathbb{P}^2$.

The unique positive-dimensional fiber of π_2 is $\Pi_0 = \mathbb{P}(V_5/A) \simeq \mathbb{P}^2$. To compute its normal bundle, let $F = \mathbb{P}_{Q^3_U}(V_5/\mathcal{U}_U) \times Q^3_P$. The variety $X \subset F$ is the regular zero locus of the section of $(V_5/\mathcal{U}_3) \otimes \mathcal{U}_P^\vee$ induced by φ, and on Π_0 its normal sequence is

$$0 \longrightarrow N_{\Pi_0/X} \longrightarrow (T_A Q^3_U \oplus T_{P_0} Q^3_P) \otimes \mathcal{O}_{\Pi_0} \longrightarrow A^\vee \otimes \mathcal{Q}_{\Pi_0} \longrightarrow 0.$$

The middle term identifies with $\mathrm{Hom}(A, V_5/A) \otimes \mathcal{O}$ via the difference of the two tangent maps, which is an isomorphism by transversality. The last map is the tautological quotient associated with $0 \to \mathcal{O}_{\Pi_0}(-1) \to (V_5/A) \otimes \mathcal{O}_{\Pi_0} \to \mathcal{Q}_{\Pi_0} \to 0$. Thus $N_{\Pi_0/X} \simeq A^\vee \otimes \mathcal{O}_{\Pi_0}(-1) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. Moreover, $b|_{\Pi_0} = h|_{\Pi_0} = 0$ and $L|_{\Pi_0} = \mathcal{O}_{\mathbb{P}^2}(1)$, so $-K_X$ has degree one on its lines: π_2 is a K-negative extremal small contraction.

Description of π_3. Interchanging (K_4, ν) with (Φ_4, ω) gives

$$Y_3 \simeq \mathscr{Z}\big(\mathrm{Fl}(2,3,V_5), \mathcal{U}_2^\vee \oplus \det \mathcal{U}_2^\vee \oplus \det \mathcal{U}_3^\vee\big) \simeq \mathrm{Bl}_{\Pi_K} W_5.$$

For the rank-three quotient $V_5 \otimes \mathcal{O}/\mathcal{U}_U$ on Q^3_U, the relative projective bundle of three-planes containing U_2 has projection p_3 and satisfies

$$p_{3*}\big(\mathcal{O}(1) \otimes p_3^* \mathcal{O}_{Q^3_U}(1)\big) = (V_5 \otimes \mathcal{O}/\mathcal{U}_U)^\vee(1), \qquad R^i p_{3*} = 0 \quad (i > 0).$$

The induced equation is the composite $\bigwedge^3 \mathcal{U}_3 \to \bigwedge^3 V_5 \xrightarrow{\mu \wedge \omega} \mathcal{O}$, as for π_1. The center of π_3 is the strict transform $S_3 = \mathrm{Bl}_{[K_3]} \Pi_\Phi \simeq \mathbb{F}_1$, whence $X \simeq \mathrm{Bl}_{S_3} Y_3$ and $E_3 = L - h$.

The two blow-ups can be performed in either order: locally at $[K_3]$ the ideals of the transverse planes are (x_1, x_2) and (x_3, x_4), and the resulting fiber product is the blow-up of their product, equal to the ideal of $\Pi_K \cup \Pi_\Phi$. Hence $X \simeq \mathrm{Bl}_{\Pi_K \cup \Pi_\Phi} W_5$ with fiber $\mathbb{P}^1 \times \mathbb{P}^1$ over $[K_3]$.

Finally, if f_1, f_3 are lines in the fibers of the two blow-ups and f_2 a line in Π_0, then

	f_1	f_2	f_3
b	1	0	0
L	0	1	0
h	0	0	1

The classes b, L, h are nef and the f_i are effective. Since $\rho(X) = 3$, these dual bases prove the asserted descriptions of the entire nef and Mori cones, and identify all three extremal rays. Rationality follows from the generically $\mathbb{P}^1$-bundle $Y_1 \to Q^3_P$, since Q^3_P is rational. ⋄

References

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