Fano 3-310-67-a. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Fl}(1,3,5),\ (V_5/\mathcal{U}_3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes (\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee\big)$.

Invariants. $h^0(-K) = 67, \quad (-K)^4 = 310, \quad h^{1,1} = 3, \quad h^{2,2} = 5.$

Description. Regard $\mathrm{Fl}(1,3,5)$ as the incidence variety in $\mathbb{P}(V_5) \times \mathrm{Gr}(3, V_5)$, preceded by the factor $\mathbb{P}^2$; π_i forgets the i-th factor. Put $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))|_X$, $a = c_1(\mathcal{U}_1^\vee)|_X$ and $L = c_1(\det \mathcal{U}_3^\vee)|_X$. Let $\ell \subset \mathbb{P}^4$ be a line and let $T \simeq \mathbb{F}_1 \subset \mathbb{P}^4$ be a smooth cubic scroll containing ℓ as its directrix.

(1) $\pi_1 \colon X \to W = \mathrm{Bl}_\ell \mathbb{P}^4$ is the blow-up of the strict transform $\widetilde{T} \simeq \mathbb{F}_1$ of T. Thus $X \simeq \mathrm{Bl}_{\widetilde{T}}(\mathrm{Bl}_\ell \mathbb{P}^4)$.
(2) $\pi_2 \colon X \to \mathbb{P}^2 \times \mathbb{P}^2$ is the blow-up of a smooth rational curve C of bidegree $(1,2)$. The target is the Fano fourfold `2-486-100-a`.
(3) $\pi_3 \colon X \to Z \subset \mathbb{P}^2 \times \mathbb{P}^4$ is $\mathrm{Bl}_S Z$, where $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and

$$Z = \mathscr{Z}_{\mathbb{P}^2 \times \mathbb{P}^4}\big(\mathcal{O}(1,1)^{\oplus 2}\big) \simeq \mathrm{Bl}_T \mathbb{P}^4$$

is also a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$. The target has the invariants of the Fano family `2-351-75-a`.

- $\mathrm{Nef}(X) = \langle h, a, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle [f_1], [f_2], [f_3]\rangle$ are simplicial, where f_i is a line in an exceptional fibre of π_i. The two-dimensional faces are contracted by the projections to $\mathbb{P}^2$, $\mathbb{P}^4$ and $\mathbb{P}^2$, respectively.
- X is rational.

Semiorthogonal decompositions. Orlov's formula [2], applied to the three blow-ups and, when appropriate, to their targets, gives

$$\begin{aligned} D^{\mathrm{b}}(X) &= \big\langle D^{\mathrm{b}}(\mathbb{F}_1), D^{\mathrm{b}}(\mathbb{P}^1), D^{\mathrm{b}}(\mathbb{P}^1), D^{\mathrm{b}}(\mathbb{P}^4)\big\rangle, \\ D^{\mathrm{b}}(X) &= \big\langle D^{\mathrm{b}}(C), D^{\mathrm{b}}(C), D^{\mathrm{b}}(\mathbb{P}^2 \times \mathbb{P}^2)\big\rangle, \\ D^{\mathrm{b}}(X) &= \big\langle D^{\mathrm{b}}(S), D^{\mathrm{b}}(\mathbb{F}_1), D^{\mathrm{b}}(\mathbb{P}^4)\big\rangle. \end{aligned}$$

For any of these blow-ups $\phi \colon X' \to Y$ with exceptional divisor $j \colon E \hookrightarrow X'$ and projection $p \colon E \to B$ to its smooth centre, the embeddings of the exceptional components are $j_*(p^*(-) \otimes \mathcal{O}_E(-k))$, $1 \le k \le \mathrm{codim}_Y B - 1$, in decreasing order of k; the target component is embedded by ϕ^*. For the iterated decompositions, these embeddings are composed with the corresponding pullbacks.

Explanation. Let A be a three-dimensional vector space, so that the first factor is $\mathbb{P}(A)$. The two sections of $V_5/\mathcal{U}_3$ give independent vectors $v_1, v_2 \in V_5$, and their common vanishing imposes $P = \langle v_1, v_2\rangle \subset U_3$. The remaining section is a net $\omega \colon A \to \bigwedge^2 V_5^\vee$; at $([z],[u],U_3)$ its vanishing means $\omega_z(u, U_3) = 0$. Writing $R = V_5/P$, the condition $P \subset U_3$ identifies the third factor with $\mathbb{P}(R) \simeq \mathbb{P}^2$. The resulting incidence

$$W = \{([u], U_3) : [u] \in \mathbb{P}(U_3),\ P \subset U_3\} \subset \mathbb{P}(V_5) \times \mathbb{P}(R)$$

is $\mathrm{Bl}_\ell \mathbb{P}(V_5)$, with $\ell = \mathbb{P}(P)$: indeed it is the zero locus of the universal quotient-bundle section defining the graph of the projection from ℓ, as in the Cayley trick [1, §2]. Its natural divisor classes satisfy $L = a - E_\ell$. Adjunction gives $-K_X = h + 2a + L$.

Description of π_1. The projection forgetting z maps onto W. On W, put $\mathcal{E} = (\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee$. The net induces a morphism of bundles

$$\varphi \colon A \otimes \mathcal{O}_W \longrightarrow \mathcal{E}, \qquad X = \mathbb{P}_W(\ker \varphi)$$

where the latter notation denotes the incidence of lines in the fibrewise kernel. Since $\mathrm{rk}\,\mathcal{E} = 2$, the general fibre is a point and the jumping locus is $D_1(\varphi)$. The bundle $\mathcal{E}$ is globally generated, and a general φ is transverse to the rank strata. The locus $D_0(\varphi)$ is empty and $D_1(\varphi)$ is a smooth surface of the expected codimension two. More precisely,

$$c_1(\mathcal{E}) = a + L, \qquad c_2(\mathcal{E}) = a^2, \qquad [D_1(\varphi)] = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 2aL + L^2.$$

Off ℓ, the condition $\operatorname{rk}\varphi \le 1$ becomes the rank-one condition on the 2×3 matrix of linear forms $\omega_\bullet(u,v_1),\omega_\bullet(u,v_2)$. To recognize its image in $\mathbb{P}(V_5)$, consider, over $\mathbb{P}(P)\simeq\mathbb{P}^1$, the surjective morphism

$$V_5\otimes\mathcal{O}_{\mathbb{P}^1}\longrightarrow A^\vee\otimes\mathcal{O}_{\mathbb{P}^1}(1),\qquad u\longmapsto\omega_\bullet(\nu,sv_1+tv_2).$$

Its rank-two kernel $\mathcal{K}$ contains the tautological $\mathcal{O}(-1)$ and has determinant $\mathcal{O}(-3)$; hence $\mathcal{K}/\mathcal{O}(-1)=\mathcal{O}(-2)$. The extension splits, and $\mathcal{K}=\mathcal{O}(-1)\oplus\mathcal{O}(-2)$. The induced map $\mathbb{P}_{\mathbb{P}^1}(\mathcal{K})\to\mathbb{P}(V_5)$ is the embedding of the smooth cubic scroll $T\simeq\mathbb{F}_1$, with ℓ as its section of degree one. Over each point of ℓ, the rank-one condition on W determines a unique point of the exceptional $\mathbb{P}^2$; hence $D_1(\varphi)=\widetilde{T}$ scheme-theoretically. Locally along this regular codimension-two centre, the matrix of φ reduces to $\left(\begin{smallmatrix}1&0&0\\0&f&g\end{smallmatrix}\right)$, with (f,g) generating its ideal. Its kernel incidence is precisely the blow-up of (f,g), proving $X=\operatorname{Bl}_{\widetilde{T}}W$. Since $-K_W=3a+2L$, its exceptional divisor is $E_1=a+L-h$.

Description of π_2. Let $B=\mathbb{P}(A)\times\mathbb{P}(R)\simeq\mathbb{P}^2\times\mathbb{P}^2$. Over B the intermediate incidence is the projective bundle $M=\mathbb{P}_B(\mathcal{O}^{\oplus 2}\oplus\mathcal{O}(0,-1))$, and the remaining section lies in $p^*\mathcal{O}_B(1,0)\otimes(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee$. If $p\colon M\to B$ is the projection, the relative Euler sequence gives

$$p_*\big((\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee\big)=\bigwedge^2(\mathcal{O}_B^{\oplus 2}\oplus\mathcal{O}_B(0,-1))^\vee,\qquad R^ip_*=0\quad(i>0).$$

Thus the induced section on B is

$$\sigma\in H^0\big(B,\mathcal{O}(1,0)\oplus\mathcal{O}(1,1)^{\oplus 2}\big).$$

The three components are independent, as is clear from the splitting of $\bigwedge^2 V_5^\vee$ relative to $V_5=P\oplus R$. Its regular zero scheme is $C=Z_B(l,q_1,q_2)$, with l of degree $(1,0)$ and q_1,q_2 of degree $(1,1)$. Over the line $L_0=Z(l)\simeq\mathbb{P}^1$, the two remaining equations give a surjective bundle map

$$R\otimes\mathcal{O}_{L_0}\longrightarrow\mathcal{O}_{L_0}(1)^{\oplus 2},\qquad\ker\simeq\mathcal{O}_{L_0}(-2).$$

Hence C is the graph of the induced conic embedding $L_0\to\mathbb{P}(R)$, and is smooth of bidegree $(1,2)$. For fixed (z,U_3), the fibre of $X\to B$ is the projectivized kernel of the alternating form $\omega_z|_{U_3}$: it is a point outside C and the whole $\mathbb{P}(U_3)=\mathbb{P}^2$ over C. Locally the kernel-incidence equations are the incidence equations of the three generators (l,q_1,q_2) of $\mathcal{I}_C$; equivalently, they are the closure of the graph of $[q_2:-q_1:l]$. Thus $X=\operatorname{Bl}_C B$ scheme-theoretically. The blow-up formula gives $-K_X=3h+3L-2E_2$, whence $E_2=h+L-a$.

Description of π_3. The projection forgetting U_3 has image

$$Z=\big\{([z],[u]):\omega_z(u,v_1)=\omega_z(u,v_2)=0\big\}\subset\mathbb{P}(A)\times\mathbb{P}(V_5).$$

These are two divisors of degree $(1,1)$. Equivalently, over $\mathbb{P}(A)$ they define the kernel of the morphism

$$\psi\colon V_5\otimes\mathcal{O}_{\mathbb{P}(A)}\longrightarrow\mathcal{O}_{\mathbb{P}(A)}(1)^{\oplus 2}.$$

This map is everywhere surjective for general ω, since its rank-one locus has expected codimension four on $\mathbb{P}^2$. Consequently, $Z=\mathbb{P}_{\mathbb{P}^2}(\ker\psi)$ is a smooth $\mathbb{P}^2$-bundle. Its projection to $\mathbb{P}(V_5)$ is the kernel incidence of the same 2×3 matrix studied above; the rank-one locus is T and the rank-zero locus is empty. The same determinantal blow-up argument proves $Z\simeq\operatorname{Bl}_T\mathbb{P}^4$.

The locus of Z over ℓ is $S=L_0\times\ell\simeq\mathbb{P}^1\times\mathbb{P}^1$. For $u\notin\ell$ there is a unique $U_3=P+\langle u\rangle$, so $X\to Z$ is an isomorphism there. If $[u]=[sv_1+tv_2]\in\ell$ and $z\in L_0$, the possible $U_3=P+\langle y\rangle$ are parametrized by the line $\omega_z(u,y)=0$ in $\mathbb{P}(R)$. To identify the contraction scheme-theoretically, write locally $u=sv_1+tv_2+x$, $x\in R$. The defining equations of Z are

$$\omega_z(x,v_1)-t\,\omega_z(v_1,v_2)=0,\qquad\omega_z(x,v_2)+s\,\omega_z(v_1,v_2)=0.$$

Eliminating $\omega_z(v_1,v_2)$ leaves the single linear equation $\omega_z(x,sv_1+tv_2)=0$ in the three normal coordinates x. It is nonzero along S for a general net, so the ideal of S in Z is locally generated by two independent coordinates of x. The additional choice of U_3 records their projective direction, which is exactly the blow-up of this regular ideal. Therefore $X=\operatorname{Bl}_S Z$. Adjunction yields $-K_Z=h+3a$ and $E_3=a-L$. Moreover $(-K_Z)^4=351$ and $h^0(-K_Z)=75$.

Nef and Mori cones. The classes h, a, L are nef as pullbacks of hyperplane classes from the three factors. Let f_1 be a fibre of the exceptional divisor of π_1, f_2 a line in a $\mathbb{P}^2$-fibre of that of π_2, and f_3 a fibre of that of π_3. Their intersections are

	f_1	f_2	f_3
h	1	0	0
a	0	1	0
L	0	0	1
$E_1 = a + L - h$	-1	1	1
$E_2 = h + L - a$	1	-1	1
$E_3 = a - L$	0	1	-1
$-K_X = h + 2a + L$	1	2	1

The blow-up description of π_2 gives $\rho(X) = 3$. Thus h, a, L and f_1, f_2, f_3 are dual numerical bases; the nef classes and their effective dual curves give the full simplicial cones stated above. The supporting classes h, a, L contract respectively the faces $\langle f_2, f_3\rangle$, $\langle f_1, f_3\rangle$ and $\langle f_1, f_2\rangle$. In particular, the two maps to $\mathbb{P}^2$ have general fibre $\mathbb{P}^2$, but are not everywhere projective bundles: over the corresponding image of C, a fibre is the union of $\mathrm{Bl}_p \mathbb{P}^2$ and $\mathbb{P}^2$, meeting along a line. Finally, the blow-up description over $\mathbb{P}^4$ shows that X is rational. ◇

References

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