Fano 3-321-69-a. $\mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Fl}(2,3,6),\ V_6/\mathcal{U}_3 \oplus (\det \mathcal{U}_3^\vee)^{\oplus 2} \oplus \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}_2^\vee\big)$, with $\mathrm{Fl}(2,3,6) \subset \mathrm{Gr}(2,6) \times \mathrm{Gr}(3,6)$.

Invariants. $h^0(-K) = 69,\ (-K)^4 = 321,\ h^{1,1} = 3,\ h^{2,1} = 0,\ h^{3,1} = 0,\ h^{2,2} = 5,\ -\chi(T) = -7$.

Description.

(1) $\pi_1 \colon X \to \mathrm{Gr}(2,6) \times \mathrm{Gr}(3,6)$ is an extremal small contraction: it contracts a plane with normal bundle $\mathcal{O}(-1)^{\oplus 2}$ to a point, and is an isomorphism elsewhere.
(2) $\pi_2 \colon X \to \mathbb{P}^2 \times \mathrm{Gr}(3,6)$ is an extremal small contraction: it contracts a second plane with normal bundle $\mathcal{O}(-1)^{\oplus 2}$ to a point, and is an isomorphism elsewhere.
(3) $\pi_3 \colon X \to \mathbb{P}^2 \times \mathrm{Gr}(2,6)$ is an extremal divisorial contraction onto a special member Z of the family 2-352-75-b, the blow-up of Z along a plane. The fibration $Z \to \mathbb{P}^2$ in quadric surfaces has a fiber which is the union of this plane and of the image of the plane contracted by π_2, meeting along a line.

- h, b, L denote the pullbacks of the hyperplane class of $\mathbb{P}^2$ and of the Plücker classes of $\mathrm{Gr}(2,6)$ and $\mathrm{Gr}(3,6)$.
- $\mathrm{Nef}(X) = \langle h, b, L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_3 and [Orl]: $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2), \pi_3^* \mathrm{D}^{\mathrm{b}}(Z)\rangle$.

Explanation.

Write $\mathbb{P}^2 = \mathbb{P}(V_3)$ and V_6 for the space of $\mathrm{Fl}(2,3,6)$, so that X consists of triples (z, U_2, U_3) with $U_2 \subset U_3$. The section of $V_6/\mathcal{U}_3$ is a vector v_0, and imposes $v_0 \in U_3$. The two sections of $\det \mathcal{U}_3^\vee$ are Plücker linear forms $H_1, H_2 \in \bigwedge^3 V_6^\vee$. The section of $\mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}_2^\vee$ is a general injection $\psi \colon V_3^\vee \to V_6^\vee$, and imposes $U_2 \subset N_z := \psi(z^\perp)^\perp$, a 4-dimensional space; the spaces N_z all contain $N_0 = \psi(V_3^\vee)^\perp$, of dimension 3. There is a unique z_0 with $v_0 \in N_{z_0}$. Put $h_i = H_i(v_0, \cdot, \cdot) \in \bigwedge^2 V_6^\vee$; for $v_0 \notin U_2$, the condition $U_2 + \langle v_0\rangle \in H_i$ reads $h_i|_{U_2} = 0$.
Description of π_1. The fiber of π_1 over (U_2, U_3) is $\{z : \psi(z^\perp)|_{U_2} = 0\}$: empty if $V_3^\vee \to U_2^\vee$ has rank 2, a reduced point if it has rank 1, and $\mathbb{P}^2$ if $U_2 \subset N_0$. In the last case $U_3 = U_2 + \langle v_0\rangle$, and the two conditions $h_i|_{U_2} = 0$ are linear on $\mathrm{Gr}(2, N_0) \simeq \mathbb{P}^2$; they define a single U_2^0. So π_1 contracts exactly the plane $\Pi_1 = \mathbb{P}^2 \times \{(U_2^0, U_2^0 + \langle v_0\rangle)\}$. Its image is cut out, in the smooth sixfold of pairs (U_2, U_3) with $v_0 \in U_3 \in H_1 \cap H_2$, by the determinantal condition $\dim(U_2 \cap N_0) \geq 1$ of the expected codimension 2; it is Cohen–Macaulay and smooth outside one point, hence normal. A line $f_1 \subset \Pi_1$ has $h \cdot f_1 = 1$ and $b \cdot f_1 = L \cdot f_1 = 0$, and the nef class $b + L$ vanishes exactly on the curves of Π_1. So π_1 is the contraction of an extremal ray, which is K_X-negative since X is Fano, and $N_{\Pi_1/X} \simeq \mathcal{O}(-1)^{\oplus 2}$ by [Kaw].

Description of π_2. Over (z, U_3), the plane U_2 must lie in $U_3 \cap N_z$; it is determined when this intersection has dimension 2, and runs over $\mathrm{Gr}(2, U_3) \simeq \mathbb{P}^2$ when $U_3 \subset N_z$. In the last case $v_0 \in N_z$ forces $z = z_0$, and the 3-spaces U_3 with $v_0 \in U_3 \subset N_{z_0}$ form $\mathrm{Gr}(2, N_{z_0}/v_0) \simeq \mathbb{P}^2$, on which the two conditions $U_3 \in H_i$ are linear: they define a single U_3^0. So π_2 contracts exactly the plane $\Pi_2 = \{z_0\} \times \mathrm{Gr}(2, U_3^0)$. Its image is cut out, in the smooth sixfold $\mathbb{P}^2 \times \{U_3 : v_0 \in U_3 \in H_1 \cap H_2\}$, by the determinantal condition that $U_3 \to V_6/N_z$ has rank at most 1, of the expected codimension 2; as before it is normal. A line $f_2 \subset \Pi_2$ has $b \cdot f_2 = 1$ and $h \cdot f_2 = L \cdot f_2 = 0$, the nef class $h + L$ vanishes exactly on the curves of Π_2, and [Kaw] gives $N_{\Pi_2/X} \simeq \mathcal{O}(-1)^{\oplus 2}$.

Description of π_3. If $v_0 \notin U_2$, then $U_3 = U_2 + \langle v_0 \rangle$ is determined, and the conditions are $h_1|_{U_2} = h_2|_{U_2} = 0$. So the image of π_3 is

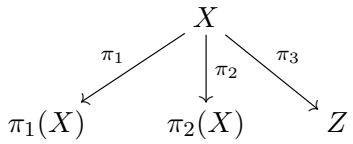
$$Z = \{(z, U_2) : U_2 \subset N_z,\ h_1|_{U_2} = h_2|_{U_2} = 0\} = \mathscr{Z}\big(\mathbb{P}^2 \times \mathrm{Gr}(2,6), \mathcal{Q}_{\mathbb{P}^2} \boxtimes \mathcal{U}^\vee \oplus \mathcal{O}(1)^{\oplus 2}\big),$$

the model of 2-352-75-b, for a special section: the 2-forms h_i have v_0 in their kernel. By [BFMT, Proposition 3.3], $\{(z, U_2) : U_2 \subset N_z\}$ is a $\mathrm{Gr}(2,4)$-bundle over $\mathbb{P}^2$, so $Z \to \mathbb{P}^2$ is a fibration in quadric surfaces. If $v_0 \in U_2$, then $z = z_0$ and U_2 runs over the plane $S = \{U_2 : v_0 \in U_2 \subset N_{z_0}\}$, on which h_1, h_2 vanish identically; over each such U_2, the 3-spaces $U_3 \supset U_2$ with $U_3 \in H_1 \cap H_2$ form a line. So π_3 is birational, with line fibers f_3 over S, and $L \cdot f_3 = 1$, $h \cdot f_3 = b \cdot f_3 = 0$; the nef class $h + b$ vanishes exactly on them, and [And, Thm. 2.3] gives that Z is smooth and $X \simeq \mathrm{Bl}_S Z$. The fiber of $Z \to \mathbb{P}^2$ over z_0 is the quadric surface $\mathrm{Gr}(2, N_{z_0}) \cap \{h_1 = h_2 = 0\}$, which contains the plane S; it is therefore the union of S and the plane $S' = \mathrm{Gr}(2, U_3^0)$, meeting along a line, and the strict transform of S' is Π_2.

Cones and rationality. The classes h, b, L are nef, and the curves f_1, f_2, f_3 form the dual basis:

	f_1	f_2	f_3
h	1	0	0
b	0	1	0
L	0	0	1

Hence $\mathrm{Nef}(X) = \langle h, b, L \rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, f_2, f_3 \rangle$ are dual simplicial cones. The composite $X \to \mathbb{P}^2$ is a fibration in quadric surfaces, and the plane Π_1 is a section of it; so its generic fiber is a quadric surface over $\mathbb{C}(\mathbb{P}^2)$ with a rational point, hence rational, and X is rational.



◇

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