Fano 4-215-49-b. $\mathscr{Z}\big(\mathbb{P}^1\times\mathbb{P}^2\times\mathrm{Fl}(1,3,5),\ (V_5/\mathcal{U}_3)^{\oplus 2}\oplus\mathcal{O}_{\mathbb{P}^2}(1)\boxtimes(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee\oplus\mathcal{O}_{\mathbb{P}^1}(1)\boxtimes\mathcal{U}_1^\vee\big)$, with $\mathrm{Fl}(1,3,5)\subset\mathbb{P}^4\times\mathrm{Gr}(3,5)$.

Invariants. $h^0(-K)=49,\ (-K)^4=215,\ h^{1,1}=4,\ h^{2,1}=0,\ h^{3,1}=0,\ h^{2,2}=9,\ -\chi(T)=0$.

Description.

(1) $\pi_1\colon X\to\mathbb{P}^2\times\mathbb{P}^4\times\mathrm{Gr}(3,5)$ is an extremal divisorial contraction onto a fourfold of the family 3-310-67-a, the blow-up of it along a sextic del Pezzo surface.
(2) $\pi_2\colon X\to\mathbb{P}^1\times\mathbb{P}^4\times\mathrm{Gr}(3,5)$ is an extremal divisorial contraction onto the blow-up of $\mathbb{P}^4$ along a line ℓ and a disjoint plane (3-400-84-b), the blow-up of it along the strict transform of a cubic scroll with directrix ℓ, a surface $\mathbb{F}_1$ blown up at 3 points.
(3) $\pi_3\colon X\to\mathbb{P}^1\times\mathbb{P}^2\times\mathrm{Gr}(3,5)$ is an extremal divisorial contraction onto a smooth divisor of tridegree $(1,1,1)$ in $\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2$ (3-288-63-a), the blow-up of it along $\mathbb{P}^1\times C$, where $C\subset\mathbb{P}^2\times\mathbb{P}^2$ is a smooth rational curve of bidegree $(1,2)$.
(4) $\pi_4\colon X\to\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^4$ is an extremal divisorial contraction onto the blow-up of $\mathbb{P}^4$ along a cubic scroll and then along the preimage of a general plane (3-256-57-g), the blow-up of it along a surface $\mathbb{P}^1\times\mathbb{P}^1$.

- h_1,h_2,h_3,L denote the pullbacks of the hyperplane classes of $\mathbb{P}^1$, $\mathbb{P}^2$, $\mathbb{P}^4$ and of the Plücker class.
- $\mathrm{Nef}(X)=\langle h_1,h_2,h_3,L\rangle$ is simplicial.
- X is rational.

Semiorthogonal decompositions.

- From π_3 and [Orl], since the target of π_3 is the blow-up of $\mathbb{P}^2\times\mathbb{P}^2$ along a sextic del Pezzo surface, $\mathrm{D}^{\mathrm{b}}(X)$ has a full exceptional collection of length $4+6+9=19$.

Explanation.

Write $\mathbb{P}^2=\mathbb{P}(A_3)$, $\mathbb{P}^4=\mathbb{P}(V_5)$ and $\mathrm{Gr}(3,5)=\mathrm{Gr}(3,V_5)$, so that X consists of quadruples (t,z,u,U_3) with $u\subset U_3$. The two sections of $V_5/\mathcal{U}_3$ are vectors v_1,v_2 and impose $P=\langle v_1,v_2\rangle\subset U_3$; let $\ell=\mathbb{P}(P)$. The section of $\mathcal{O}(1)\boxtimes(\mathcal{U}_3/\mathcal{U}_1)^\vee\otimes\mathcal{U}_1^\vee$ is a general net $\omega\colon A_3\to\bigwedge^2V_5^\vee$, and imposes $\omega_z(u,U_3)=0$. The section of $\mathcal{O}(1)\boxtimes\mathcal{U}_1^\vee$ is a general pencil of linear forms λ_t, and imposes $\lambda_t(u)=0$; its base locus is a plane $\Pi\subset\mathbb{P}^4$, disjoint from ℓ. Thus $X=\{\lambda_t(u)=0\}\subset\mathbb{P}^1\times X_0$, where X_0 is the zero locus of the first three summands.

For $w\in P$, the conditions $\omega_\bullet(u,w)=0\in A_3^\vee$ define a line $L_w\ni w$ of $\mathbb{P}^4$, and two such lines are disjoint, since $\omega_\bullet(u,v_1)=\omega_\bullet(u,v_2)=0$ are six general conditions. Hence the union T of the lines L_w is a smooth cubic scroll with directrix ℓ; it is the locus where the 2×3 matrix of linear forms $(\omega_\bullet(u,v_1),\omega_\bullet(u,v_2))$ has rank at most one, of class $3h_3^2$ [Ful, Thm. 14.4]. The plane Π meets T transversally in 3 points, which are not collinear: a line through three of them would lie both in T, which is cut out by quadrics, and in Π.

Description of π_1. Since $\mathcal{O}(h_3)$ is globally generated on X_0, [BFMT, Lemma 3.1] gives $X \simeq \mathrm{Bl}_{S_1} X_0$, where S_1 is the preimage of Π under $X_0 \to \mathbb{P}^4$; X_0 is the model of 3-310-67-a. For $u \notin \ell$ we have $U_3 = P + \langle u \rangle$, and z is cut out by the two linear conditions $\omega_z(u, v_1) = \omega_z(u, v_2) = 0$: it is unique for $u \notin T$, and runs over a line for $u \in T$. As Π misses ℓ, the smooth surface S_1 maps birationally onto Π, contracting exactly one line over each point of $\Pi \cap T$. Hence S_1 is Π blown up at three non-collinear points, a sextic del Pezzo surface. A fiber f_1 of π_1 has $h_1 \cdot f_1 = 1$ and $h_2 \cdot f_1 = h_3 \cdot f_1 = L \cdot f_1 = 0$.

Description of π_2. The vectors v_1, v_2 cut out $W = \{(u, U_3) : P \subset U_3\} \simeq \mathrm{Bl}_\ell \mathbb{P}^4$ in $\mathrm{Fl}(1, 3, 5)$, and [BFMT, Lemma 3.1] identifies the image of π_2 with $W' = \mathrm{Bl}_\Pi W = \mathrm{Bl}_{\ell \sqcup \Pi} \mathbb{P}^4$, the model of 3-400-84-b. Over $(t, u, U_3) \in W'$, the condition on z is the morphism

$$\psi \colon A_3 \otimes \mathcal{O} \longrightarrow (\mathcal{U}_3/\mathcal{U}_1)^\vee \otimes \mathcal{U}_1^\vee, \qquad z \longmapsto \omega_z(u, \cdot)|_{U_3},$$

of ranks $3 \to 2$, and the fiber of π_2 is $\mathbb{P}(\ker \psi)$. The target bundle is generated by $\bigwedge^2 V_5^\vee$ and ω is general, so ψ never vanishes and drops rank along a smooth surface of W; off ℓ this is the condition $u \in T$, so this surface is the strict transform $\widetilde{T} \simeq \mathbb{F}_1$ of T. Its preimage in W' is the strict transform T' of $\widetilde{T}$, which is $\widetilde{T}$ blown up at its 3 transversal intersection points with Π. So π_2 is birational, with fibers the lines f_2 over T', and $h_2 \cdot f_2 = 1$, $h_1 \cdot f_2 = h_3 \cdot f_2 = L \cdot f_2 = 0$.

Description of π_3. The planes $U_3 \supset P$ form $\mathbb{P}(V_5/P) \simeq \mathbb{P}^2$, and $U_3 = P + \langle y \rangle$. Since $\bigwedge^2 U_3^\vee \simeq U_3 \otimes \det U_3^\vee$, the form $\omega_z|_{U_3}$ is a section k of $\mathcal{U}_3 \otimes \det \mathcal{U}_3^\vee(h_2)$ on $B = \mathbb{P}^2_z \times \mathbb{P}^2_y$, and the condition on u is $u \in \langle k(z, y) \rangle$. The section k vanishes on the curve

$$C = \{\omega_z(v_1, v_2) = \omega_z(v_1, y) = \omega_z(v_2, y) = 0\} \subset B.$$

The first equation defines a line $L_0 \subset \mathbb{P}^2_z$, and over L_0 the other two define a surjection $V_5/P \otimes \mathcal{O}_{L_0} \to \mathcal{O}_{L_0}(1)^{\oplus 2}$ with kernel $\mathcal{O}_{L_0}(-2)$. So C is the graph of a conic embedding of L_0, a smooth rational curve of bidegree $(1, 2)$. Off C, u is determined, and $\lambda_t(k) = 0$ is a section of $\mathcal{O}(1, 1, 1)$ on $\mathbb{P}^1 \times B$; so the image of π_3 is the divisor $D = \{\lambda_t(k(z, y)) = 0\}$, the model of 3-288-63-a. Over $\mathbb{P}^1 \times C$ the fiber is the line $\mathbb{P}(U_3) \cap \{\lambda_t = 0\}$, and elsewhere a point. A fiber f_3 has $h_3 \cdot f_3 = 1$ and $h_1 \cdot f_3 = h_2 \cdot f_3 = L \cdot f_3 = 0$. The divisor D is smooth outside the surface $\mathbb{P}^1 \times C$, hence normal.

Description of π_4. The image of π_4 is

$$Z' = \{(t, z, u) : \omega_z(u, v_1) = \omega_z(u, v_2) = 0,\ \lambda_t(u) = 0\},$$

the model of 3-256-57-g. Its projection to $\mathbb{P}^2 \times \mathbb{P}^4$ is $Z_0 = \{\omega_z(u, v_1) = \omega_z(u, v_2) = 0\}$, the incidence of the kernels of the matrix above, which is $\mathrm{Bl}_T \mathbb{P}^4$; then [BFMT, Lemma 3.1] gives $Z' = \mathrm{Bl}_{S'} Z_0$, where S' is the preimage of Π, a sextic del Pezzo surface. For $u \notin \ell$, $U_3 = P + \langle u \rangle$ is determined. For $u \in \ell$, the condition $\omega_z(u, P) = 0$ forces $z \in L_0$, then t is determined by $\lambda_t(u) = 0$ since $\ell \cap \Pi = \emptyset$, and $U_3 = P + \langle y \rangle$ with y on the line $\omega_z(u, y) = 0$. So π_4 is birational, with fibers the lines f_4 over a surface isomorphic to $L_0 \times \ell \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and $L \cdot f_4 = 1$, $h_1 \cdot f_4 = h_2 \cdot f_4 = h_3 \cdot f_4 = 0$.

Contractions and cones. The classes h_1, h_2, h_3, L are nef, and the curves $f_1, \dots, f_4$ form the dual basis:

	f_1	f_2	f_3	f_4
h_1	1	0	0	0
h_2	0	1	0	0
h_3	0	0	1	0
L	0	0	0	1

Hence $\mathrm{Nef}(X) = \langle h_1, h_2, h_3, L\rangle$ and $\overline{\mathrm{NE}}(X) = \langle f_1, \dots, f_4\rangle$ are dual simplicial cones. Each π_i is supported by the sum of the three nef classes other than the i-th one, contracts exactly the curves of class proportional to f_i, and has fibers of dimension at most one; its image is normal. Since X is Fano, π_i is the contraction of a K_X-negative extremal ray, and [And, Thm. 2.3] shows that it is the blow-up of a smooth surface in a smooth fourfold. In particular the four targets are smooth, and the centers are the surfaces described above. For π_3, writing $D = \{t_0 s_0 + t_1 s_1 = 0\}$ with $s_0, s_1 \in H^0(B, \mathcal{O}(1,1))$, the smoothness of D shows that $\Sigma = \{s_0 = s_1 = 0\}$ is a smooth complete intersection of two divisors of bidegree $(1,1)$, a sextic del Pezzo surface, and $D \simeq \mathrm{Bl}_\Sigma(\mathbb{P}^2 \times \mathbb{P}^2)$ by [BFMT, Lemma 3.1]. In particular X is rational.

◇

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