Fano 6-172-40-a. $\mathscr{Z}\big(\mathrm{Gr}(2,4)\times\mathrm{Gr}(2,5),\mathcal{U}^\vee\boxtimes\mathcal{Q}^\vee(1)\big)$.

Invariants. $h^0(-K)=40,\quad (-K)^4=172,\quad h^{1,1}=6,\quad h^{2,2}=17,\quad -\chi(T)=0.$

Description. **This Fano fourfold was previously studied in [3].**

Let $Q=\mathrm{Gr}(2,V_4)\simeq\mathbb{Q}^4$ be the Fano fourfold 1-512-105-a, and let π_i omit the i-th factor. A general defining section gives an injection $\varphi: V_4\hookrightarrow\bigwedge^2 V_5$ whose projectivized image meets $\mathrm{Gr}(2,V_5)$ in five reduced points $[\bigwedge^2 P_i]$. Write $\Pi_i=\{A\in Q:\varphi^{-1}(\bigwedge^2 P_i)\subset A\}\simeq\mathbb{P}^2$; these planes meet pairwise at the ten points p_{ij}.

(1) $\pi_1\colon X\to Y\subset\mathrm{Gr}(2,V_5)$ is a K-negative small contraction onto a determinantal fourfold of degree 12 with ten singular points, locally cones over $\mathbb{P}^1\times\mathbb{P}^2$. Its exceptional locus consists of ten disjoint planes $\mathbb{P}^2$, each with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

(2) $\pi_2\colon X\to Q$ is a birational contraction onto the smooth quadric, with exceptional divisors over the five planes Π_i. Its fibers are points away from $\bigcup_i\Pi_i$, are $\mathbb{P}^1$ over a general point of each Π_i, and are $\mathbb{P}^1\times\mathbb{P}^1$ over p_{ij}. The blow-up-and-contraction construction is described in [3, § 4.4].

Let $H=\mathcal{O}(1,0)|_X$, $L=\mathcal{O}(0,1)|_X$, and let E_i be the divisors over Π_i.

- $-K_X=H+L=4H-\sum_i E_i$ and $L=3H-\sum_i E_i$.
- $\mathrm{Nef}(X)=\big\{aH-\sum_i b_iE_i:\ b_i\ge 0,\ a\ge b_i+b_j+b_k \text{ for } i<j<k\big\}$. It has the twelve generators H, $H-E_i$, $3H-\sum_{j\ne i}E_j$, and L; $\overline{\mathrm{NE}}(X)$ has fifteen extremal rays. Neither cone is simplicial.
- X is rational.

Semiorthogonal decompositions. The factorization of π_2 into elementary blow-ups, with successive centers S_k obtained by blowing up $\mathbb{P}^2$ at k distinct points ($0\le k\le 4$), gives by Orlov's formula [4]

$$\mathrm{D^b}(X)=\big\langle\mathrm{D^b}(S_4),\mathrm{D^b}(S_3),\mathrm{D^b}(S_2),\mathrm{D^b}(S_1),\mathrm{D^b}(S_0),\mathrm{D^b}(Q)\big\rangle.$$

The embeddings are successive pullbacks of the standard blow-up functors $j_*(p^*(-)\otimes\mathcal{O}_E(-1))$ and of the pullback from the target.

Explanation. The tautological quotient bundle $\mathcal{Q}$ has rank three on $\mathrm{Gr}(2,V_5)$, and $\mathcal{Q}^\vee(1)\simeq\bigwedge^2\mathcal{Q}$. Thus the section is determined by $\varphi\in\mathrm{Hom}(V_4,\bigwedge^2 V_5)$ and

$$X=\{(A,B):\varphi(A)\subset B\wedge V_5\}.$$

The bundle is globally generated, hence a general zero locus is smooth. Adjunction gives $-K_X=H+L$. The geometry of the two projections is studied in [3, § 4]; we recall the determinantal construction and the consequences needed here.

Description of π_1. For the ambient projection p_1 to $\mathrm{Gr}(2,V_5)$, Borel–Weil gives

$$p_{1*}(\mathcal{U}^\vee\boxtimes\bigwedge^2\mathcal{Q})=V_4^\vee\otimes\bigwedge^2\mathcal{Q},\qquad R^ip_{1*}=0\quad(i>0).$$

The section induces

$$\psi: V_4\otimes\mathcal{O}\xrightarrow{\varphi}\bigwedge^2 V_5\otimes\mathcal{O}\longrightarrow\bigwedge^2\mathcal{Q},\qquad \mathrm{rk}\,\psi:(4\to 3),$$

with kernel incidence $X=\{(A,B):A\subset\ker\psi_B\}$ and image $Y=D_2(\psi)$ scheme-theoretically. Its fiber at B is $\mathrm{Gr}(2,\ker\psi_B)$. Since $\bigwedge^2\mathcal{Q}$ is globally generated and $H^0(\bigwedge^2\mathcal{Q})=\bigwedge^2 V_5$, generic determinantal transversality gives $\mathrm{codim}\,D_2(\psi)=2$, $\mathrm{codim}\,D_1(\psi)=6$, $D_0(\psi)=\varnothing$, with smooth rank strata. Writing $c_i=c_i(\bigwedge^2\mathcal{Q})$, the Thom–Porteous formulas [2] yield

$$[Y]=c_1^2-c_2=3\sigma_{1,1}+2\sigma_2,\qquad \deg Y=12,\qquad [D_1(\psi)]=c_2^3-2c_1c_2c_3+c_3^2=10[\mathrm{pt}].$$

Hence $D_1(\psi)$ consists of ten reduced points and its fibers are $\mathrm{Gr}(2,3) \simeq \mathbb{P}^2$; elsewhere π_1 is an isomorphism. Near such a point the map is reduced to a 2×3 matrix whose six entries form local coordinates, so Y is analytically $\{\mathrm{rk}\, M \leq 1\}$, the cone over $\mathbb{P}^1 \times \mathbb{P}^2$, and its kernel incidence is the small resolution with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. These ten planes are the transforms of the ten β-planes of the Cremona–Richmond configuration [3, Propositions 4.2 and 4.7]. Since H restricts to $\mathcal{O}_{\mathbb{P}^2}(1)$ and L is trivial on them, $-K_X$ has degree one on their lines.

Description of π_2. For the ambient projection p_2 to Q, the relative section corresponds to

$$p_{2*}(\mathcal{U}^\vee \boxtimes \bigwedge^2 \mathcal{Q}) = \mathcal{U}^\vee \otimes \bigwedge^2 V_5, \qquad R^i p_{2*} = 0 \quad (i > 0),$$

with induced map

$$\mathcal{U} \hookrightarrow V_4 \otimes \mathcal{O} \xrightarrow{\varphi} \bigwedge^2 V_5 \otimes \mathcal{O}.$$

Its fiber over $A \in Q$ is the common zero locus of the two bivectors $\varphi(A)$ in $\bigwedge^2 \mathcal{Q}$. The general fiber has length $\int_{\mathrm{Gr}(2,5)} c_3(\bigwedge^2 \mathcal{Q})^2 = 1$. The five decomposable bivectors $\bigwedge^2 P_i$ determine the planes Π_i. Over p_{ij} the two equations require $B \cap P_i \neq 0$ and $B \cap P_j \neq 0$, hence the fiber is

$$\{B = \ell_i \oplus \ell_j : \ell_i \subset P_i,\ \ell_j \subset P_j\} \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

The description of the remaining fibers, and the global model obtained by blowing up the ten points p_{ij}, then the five transformed planes $\mathrm{Bl}_4\, \mathbb{P}^2$, and contracting the ten intervening divisors onto these quadric surfaces, are given in [3, Proposition 4.6 and § 4.4]. In particular π_2 is birational and the exceptional divisors E_i map to Π_i. The five ruling classes e_i of its general exceptional fibers are independent, and span the simplicial relative Mori face $H^\perp$. On them $L \cdot e_i = 1$, $H \cdot e_i = 0$, and the divisors satisfy $E_i \cdot e_j = -\delta_{ij}$. The relative canonical divisor formula therefore gives

$$K_X = \pi_2^* K_Q + \sum_i E_i, \qquad L = 3H - \sum_i E_i.$$

Each $\mathbb{R}_{\geq 0} e_i$ is an elementary K-negative divisorial ray. Its successive contractions have one-dimensional fibers and, by [1, Theorem 2.3], are blow-ups of smooth codimension-two centers. The centers map onto the corresponding Π_i and acquire one exceptional curve at each previously resolved intersection p_{ij}; thus the five successive centers, in any chosen order, are $\mathrm{Bl}_k\, \mathbb{P}^2$ for $k = 0, 1, 2, 3, 4$. This also justifies the stated applications of Orlov's formula.

Mori and nef cones. Let l_T be the class of a line in the exceptional plane of π_1 indexed by a three-element subset $T \subset \{1, \dots, 5\}$. As the plane meets precisely the divisors E_i with $i \in T$, we have

$$\big(aH - \sum_i b_i E_i\big) \cdot e_j = b_j, \qquad \big(aH - \sum_i b_i E_i\big) \cdot l_T = a - \sum_{i \in T} b_i.$$

The dual of the cone generated by these fifteen effective classes is the cone in the Description; its twelve extremal generators are H, $H - E_i$, $3H - \sum_{j \neq i} E_j$, and L. The first and last are nef by construction, while $H - E_i$ is the pullback of the linear projection of $Q \subset \mathbb{P}^5$ from Π_i, resolved by its blow-up. For $D_i = 3H - \sum_{j \neq i} E_j = L + E_i$, nefness outside E_i follows from L and the effectiveness of E_i. The elementary contraction of e_i identifies the center with $S_i \simeq \mathrm{Bl}_4\, \mathbb{P}^2$, with $H|_{S_i} = \ell$ and $E_j|_{S_i} = \varepsilon_j$ for $j \neq i$. Thus the divisor $D_i' = 3H - \sum_{j \neq i} E_j$ on the target of this contraction restricts to

$$D_i'|_{S_i} = 3\ell - \sum_{j \neq i} \varepsilon_j = -K_{S_i},$$

which is ample. Hence D_i is nef. This proves the asserted descriptions of $\mathrm{Nef}(X)$ and $\overline{\mathrm{NE}}(X)$; in particular the ten l_T and five e_i are exactly their extremal generators. The rationality of X follows from that of Q. ◇

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