Fano 8-80-21-a. $\mathscr{Z}\big(\mathrm{Gr}(2,7),\,(\mathrm{Sym}^2\mathcal{U}^\vee)^{\oplus 2}\big)$.

Invariants. $h^0(-K)=21,\quad (-K)^4=80,\quad h^{1,1}=8,\quad h^{2,2}=30,\quad -\chi(T)=4.$

Description. **This is the Küchle fourfold (b9) [1].** If $Z=Q_1\cap Q_2\subset\mathbb{P}^6$ is a smooth complete intersection of two quadrics, then

$$X\simeq F_1(Z),$$

the Fano scheme of lines contained in Z [2, 3].

- X is rational and is obtained from $\mathrm{Bl}_{p_1,\dots,p_7}\mathbb{P}^4$, for seven points in general linear position, by 22 antiflips [3, Theorem 2.5].
- The Mori cone has 64 extremal rays, all giving small contractions of planes $\mathbb{P}^2$ to points; the nef cone has 78 extremal rays and is not simplicial [4, Theorem 5.1 and § 6].

Semiorthogonal decompositions. By [3, Corollary 2.6], $\mathrm{D}^{\mathrm{b}}(X)$ admits a full exceptional collection of length 48.

Explanation. A section of $\mathrm{Sym}^2\mathcal{U}^\vee$ is the restriction to the tautological planes of a quadratic form $q\in\mathrm{Sym}^2 V_7^\vee$. Thus the two defining sections have common zero locus

$$X=\{U\in\mathrm{Gr}(2,V_7): q_1|_U=q_2|_U=0\}=F_1(Q_1\cap Q_2)$$

scheme-theoretically. For a general pencil, $Z=Q_1\cap Q_2$ is smooth, so the results of [2, 3, 4] apply. Since $\dim\mathrm{Gr}(2,7)=10$ and the defining bundle has rank six, $\dim X=4$; adjunction gives

$$-K_X=\big(7-2\cdot 3\big)\,\mathcal{O}_{\mathrm{Gr}(2,7)}(1)|_X=\mathcal{O}_{\mathrm{Gr}(2,7)}(1)|_X.$$

The birational description and the derived-category result follow from [3, Theorem 2.5 and Corollary 2.6]; the extremal contractions and both cones are described in [4, Theorem 5.1 and § 6]. ◇

References

[1] O. Küchle, *On Fano 4-folds of index 1 and homogeneous vector bundles over Grassmannians*, Math. Z. **218** (1995), 563–575.

[2] C. Casagrande, *Rank 2 quasiparabolic vector bundles on $\mathbb{P}^1$ and the variety of linear subspaces contained in two odd-dimensional quadrics*, Math. Z. **280** (2015), 981–988.

[3] A. Kuznetsov, *On Küchle varieties with Picard number greater than 1*, Izv. Math. **79** (2015), no. 4, 698–709.

[4] C. Araujo and C. Casagrande, *On the Fano variety of linear spaces contained in two odd-dimensional quadrics*, Geom. Topol. **21** (2017), 3009–3045.